Acta Math. Hungar., **152** (1) (2017), 109–113
DOI: 10.1007/s10474-017-0702-x
First published online March 20, 2017

AN UPPER ESTIMATE FOR THE DISCREPANCY OF IRRATIONAL ROTATIONS

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(Received August 8, 2016; revised January 16, 2017; accepted January 19, 2017)

Abstract. We give an upper bound for the discrepancy of irrational rotations $\{n\alpha\}$ in terms of the continued fraction expansion of α and the related Ostrowski expansion. Our result improves earlier bounds in the literature and substantially simplifies their proofs.

1. Introduction

Let $\alpha \in (0,1)$ be an irrational number and let

$$D_N(\{i\alpha\}) = \sup_{0 \le a < b \le 1} \left| \frac{1}{N} \sum_{i=1}^{N} \mathbf{I}_{[a,b)}(\{i\alpha\}) - (b-a) \right|,$$

$$D_N^*(\{i\alpha\}) = \sup_{0 < a \le 1} \left| \frac{1}{N} \sum_{i=1}^{N} \mathbf{I}_{[0,a)}(\{i\alpha\}) - a \right|$$

denote the discrepancy, resp. star discrepancy of the first N terms of the sequence $\{i\alpha\}$, $i = 1, 2, \dots$, where $\{x\}$ denotes the fractional part of a real number x. Clearly $D_N(\{i\alpha\}) \leqq 2D_N^*(\{i\alpha\})$. Estimating $D_N(\{i\alpha\})$ and $D_N^*(\{i\alpha\})$ is a classical problem of Diophantine approximation theory and had a wide literature, see e.g. [3], [9], [13], [14] and the references therein.

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Key words and phrases: rational rotation, irrational rotation, uniform distribution sequence, continued fraction, discrepancy.

Mathematics Subject Classification: primary 11K38, secondary 11K31, 11A55.

Let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}} \tag{1}$$

denote the continued fraction expansion of α with convergents $r_n = p_n/q_n$. Ostrowski [12] showed that there is a close connection between $D_N(\{i\alpha\})$, $D_N^*(\{i\alpha\})$ and the partial quotients a_j in the continued fraction expansion of α and the coefficients b_j in the expansion

$$N = \sum_{j=0}^{m} b_j\, q_j, \tag{2}$$

where m is defined by $q_m \le N < q_{m+1}$, and each coefficient b_j satisfies

$$0 \le b_0 < a_1\,, \quad 0 \le b_j \le a_{j+1}\,,\ j \ge 1\,, \quad \text{and} \quad b_{j-1} = 0 \ \text{ if } b_j = a_{j+1}\,.$$

Schoissengeier [13, Theorem 1] gave a (rather complicated) explicit formula for $D_N^*(\{i\alpha\})$ and some simpler approximation formulas, see also [3, Section 1.4.1]. A simple corollary of his results is the upper bound

$$ND_N^*(\{i\alpha\}) \ \le\ C\Big(\sum_{n\le m} a_n + b_m\Big) \tag{3}$$

with an absolute constant C; see also [3, Corollary 1.64 on p. 52]. As shown in [3], the last estimate is optimal, except for the value of the constant C. However, the arguments in [3], [13] are long and technical and no explicit value of C is given. The purpose of the present paper is to give a simple direct proof of the following variant of (3) with the constant $C = 1$.

THEOREM 1.1. *For any natural number N we have*

$$ND_N^*(\{i\alpha\}) \le \max\Big\{\sum_{n:\text{odd},\le m} a_n, \sum_{n:\text{even},\le m} a_n\Big\} + b_m + m + 1, \tag{4}$$

where m is the number in the Ostrowski expansion (2) *of N.*

Since $b_m \leqq a_{m+1}$ and $m + 1 \leqq \sum_{n\leqq m+1} a_n$, (4) implies that

$$ND_N^*(\{i\alpha\}) \le 2 \sum_{n\leqq m+1} a_n. \tag{5}$$

Most likely, the constant 2 could be improved for sufficiently large N, but due to the small values of N, to determine the best constant seems to be very difficult.

As a sharp bound for $ND_N^*(\{i\alpha\})$, relation (5) implies a number of classical discrepancy results in the literature. From the standard linear recursion for q_n it follows that q_n grows at least exponentially and thus $m = O(\log n)$. Hence if the sequence (a_j) in the expansion (1) is bounded, then $ND_N^*(\{i\alpha\}) = O(\log N)$, a classical result from the 1920's (cf. also [9, p. 125, Theorem 3.4]). The metric behavior of the sum $\sum_{k=1}^n a_k$ is well known (see e.g. [6]) and Theorem 1.1 also implies Khinchin's sharp bound

$$ND_N^*(\{i\alpha\}) = O((\log N)^{1+\varepsilon}),$$

valid for almost all α (cf. [3, Theorem 1.72, p. 63]). Naturally, (5) also implies that for any irrational $\alpha \in (0,1)$ we have $D_N^*(\{i\alpha\}) \to 0$ as $n \to \infty$, which is the original equidistribution theorem result of Bohl, Sierpinski and Weyl (1909/1910). To see this formally, let $A = \max_{1 \le j \le m} a_j$. The following estimates are easily derived (cf. [8]);

$$N \ge A\, 2^{\frac{m}{2}-1} \ (m > 1), \quad \sum_{j=1}^m a_j \le A\, m, \quad \frac{b_m}{N} < \frac{1}{q_m},$$

and thus

$$D_N^*(\{i\alpha\}) \le m\, 2^{-\frac{m}{2}+1} + \frac{m+1}{q_m} = o(1)$$

since q_m grows at least exponentially.

The asymptotic behavior of the sum $\sum_{k=1}^n a_k$ is quite interesting. From the results of [2] and [8] it follows that the maximal term of the sum influences crucially its behavior. In [1] it is shown that removing from $\sum_{k=1}^n a_k$ its largest ω_n terms where $\omega_n \to \infty$ and $\omega_n/n \to 0$, the remaining sum is asymptotically normally distributed. The influence of an isolated large a_k on the discrepancy $D_N^*(\{i\alpha\})$ was studied in [15], [16].

2. Proof of Theorem 1.1

The proof of Theorem 1.1 uses approximation of irrational rotations by rational rotations, a method utilized in Mori–Takashima ([10]) and Shimaru–Takashima ([15]) and actually, widely in the literature. One of the main ideas in [10] and [15] can be expressed as follows.

LEMMA 2.1. *For $0 < b < a_{n+1}$, any point $\{i\alpha\}$ $(i = 1, 2, \dots, bq_n)$ belongs to one sub-interval of the form, $[j/q_n, (j+1)/q_n)$, and each sub-interval contains just b points only. Moreover, if $\{i\alpha\} \in [j/q_n, (j+1)/q_n)$ for some j,*

$$\frac{j}{q_n} < \{i\alpha\} < \{(i+q_n)\alpha\} < \{(i+2q_n)\alpha\} < \cdots < \frac{j+1}{q_n},$$

in case n is even, and

$$\frac{j+1}{q_n} > \{i\alpha\} > \{(i+q_n)\alpha\} > \{(i+2q_n)\alpha\} > \cdots > \frac{j}{q_n},$$

in case n is odd.

In the Ostrowski expansion (2), let $n_j = \sum_{k=j+1}^{m} b_k q_k$. We split the sum in the definition of ND_N^* as follows:

$$\sum_{i=1}^{N} \mathbf{I}_{[0,a)}(\{i\alpha\}) - Na = \sum_{j=0}^{m} \left\{ \sum_{i=1}^{b_j q_j} \mathbf{I}_{[0,a)}(s_j + \{i\alpha\}) - b_j q_j a \right\},$$

where s_j denotes the starting point of points $\{i\alpha\}$, $i = n_j + i'$, $i' = 1, 2, \dots$, $j = m-1, \dots, 0$, that is, $s_j = \sum_{k=j+1}^{m} b_k q_k(\alpha - r_k)$, and $s_0 = 0$. Note that $q_k(\alpha - r_k) > 0$ if k is even, and that $q_k(\alpha - r_k) < 0$ if k is odd. Decompositions of the sums were originally introduced in Ostrowski [12].

Using Lemma 2.1 and the well-known inequality $r_{2n} < \alpha < r_{2n-1}$, we easily get for even $j(\leq m)$:

$$-1 \leq \left\{ \sum_{i=n_j+1}^{n_j+b_j q_j} \mathbf{I}_{[0,a)}(\{i\alpha\}) - b_j q_j a \right\} \leq b_j + 1 \leq a_{j+1} + 1,$$

and for odd $j(\leq m)$:

$$-a_{j+1} - 1 \leq -b_j - 1 \leq \left\{ \sum_{i=n_j+1}^{n_j+b_n q_n} \mathbf{I}_{[0,a)}(\{i\alpha\}) - b_j q_j a \right\} \leq 1.$$

From these relations we immediately get Theorem 1.1.

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