UPON A STATISTICAL METHOD IN THE THEORY OF DIOPHANTINE APPROXIMATIONS.

By Aurel Wintner.

Introduction.

Let

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(\lambda_n s); \qquad a_n \neq 0$$

denote a Dirichlet series possessing linearly independent real exponents λ_n and a domain (i. e. half-plane or strip) in which $f(s)$ is absolutely convergent. Let α be a real number in the interior of this domain and set

$$z = z(t) = x(t) + iy(t) = f(\alpha + it)$$

where $-\infty < t < +\infty$. The values taken by $z(t)$ are, according to Jessen,* distributed asymptotically in such a way that there exists, in the (x, y)-plane, a continuous function $D = D(x, y)$ determining the density of this distribution, i. e. the density of probability (relative frequency as $t \to \infty$) of the values taken by $z(t) = x(t) + iy(t)$. The method of Jessen is built, on the one hand, upon an integration theory in a space of infinitely many dimensions and, on the other hand, upon the Kronecker-Weyl approximation theorem.

In the present paper the treatment of the distribution problem belonging to the almost-periodic function $z(t)$ will be based upon the general statistical or momentum method, as developed, for the one-dimensional case, by the author,† and recently extended to higher spaces by Haviland.‡ It will be proven that the continuous density function D, the existence of which (i. e. Jessen's result) need not be presupposed, is related to the distribution function § ρ belonging to the real part $x(t)$ of $z(t)$ by an integral equation of the Abel type. Since ρ is explicitly known ¶ we thus obtain an analytical method

* B. Jessen, *Bidrag til Integraltheorien for Funktioner of uendelig mange Variable*, Copenhagen, 1930.

† A. Wintner, "Diophantische Approximationen und Hermitesche Matrizen. I.," *Mathematische Zeitschrift*, Vol. 30 (1929), pp. 290-319 (more particularly pp. 310-311). This paper will be referred to as *I*.

‡ E. K. Haviland, "On statistical methods in the theory of almost-periodic functions," *Proceedings of the National Academy of Sciences*, Vol. 19 (1933), May issue.

§ First introduced *loc. cit. I*.

¶ A. Wintner, "On an application of diophantine approximation to the repartition problems of dynamics," *Journal of the London Mathematical Society*, Vol. 7 (1932), for an effective control of D. With the use of Bessel functions, the application of this explicit method yields the result that $D(x, y)$ not only is everywhere continuous but also possesses derivatives of arbitrarily high order save at most at the origin $x = y = 0$, without being analytic im grossen. The question as to whether D is analytic im kleinen remains open. On the other hand, the method works just as well in the "non-analytic" case,* where the series $f(s)$ is absolutely convergent not in a domain (i. e. half-plane or strip) but only on the isolated line $s = \alpha + it$. Hence we start directly with an arbitrary almost-periodic function

$$(1) \qquad z(t) = x(t) + iy(t) = \sum_{j=1}^{\infty} r_j \exp i\lambda_j(t - t_j); \qquad r_j > 0$$

$(-\infty < t < +\infty)$ where the frequencies λ_j are supposed to be linearly independent, in which case, according to a theorem of Bohr,† of necessity

$$(2) \qquad R < +\infty \quad \text{where} \quad R = \sum_{j=1}^{\infty} r_j.$$

It may be mentioned that the ultimate reason for the occurance of the Abel integral equation reducing D to ρ lies in the fact that on account of the Laplace-Fourier transforms of D and ρ this reduction is a transformation of "planes waves" into "spherical waves."

Applications to the μ-function of Lindelöf will be given in a subsequent paper.

The Distribution of the Real Component.

The distribution function $\rho = \rho(\xi)$ of an arbitrary ‡ real-valued almost-periodic function $x(t)$ is defined for $-\infty < \xi < +\infty$ as

$$(3) \qquad \lim_{T=+\infty} \text{meas } \{x(t) \leqq \xi;\ T\}/2T$$

where $\{x(t) \leqq \xi;\ T\}$ denotes the set of all those points t for which both inequalities $x(t) \leqq \xi$, $|t| < T$ are satisfied, and meas $\{x(t) \leqq \xi;\ T\}$ is the Lebesgue measure § of this set. The limit (3) exists ¶ save for a denumerable

pp. 242-246. This paper will be referred to as *II*. Cf. also "Ueber die statistische Unabhängigkeit," *Mathematische Zeitschrift*, Vol. 36 (1933), pp. 618-629. This paper will be referred to as *III*.

* In reality the question regarding the analytic continuation of such a function $f(\alpha + it)$ does not seem to have been treated yet in the literature.

† H. Bohr, "Zur Theorie der fast-periodischen Funktionen. I.," *Acta Mathematica*, Vol. 45 (1925), p. 103.

‡ The linear independence of the frequencies is not yet supposed.

§ This is at present a Jordan content inasmuch as $x(t)$ is almost-periodic and therefore continuous.

¶ *Loc. cit. I.*

set of exceptional values $\xi = \xi_m$ which, if they exist, are always discontinuity points * of the monotone function $\rho(\xi)$. The latter is defined as the limit (3) if $\xi \neq \xi_m$ and as the arithmetical mean of $\rho(\xi + 0)$ and $\rho(\xi - 0)$ if $\xi = \xi_m$. An exceptional point ξ_m may actually exist.† On the other hand, it is possible that ξ_m is a discontinuity point of $\rho(\xi)$ without being ‡ an exceptional point ξ_m.

Now let $x(t)$ be the real part of (1), i. e. suppose that the frequencies of the almost-periodic function $x(t)$ are linearly independent. Then $\rho(\xi)$ is everywhere continuous §; hence (3) exists for every ξ. We shall see later on that all derivatives of $\rho(\xi)$ exist. Let $\rho_k(\xi)$ denote the distribution function belonging to the partial sum

(4) $$x_k(t) = \sum_{j=1}^{k} r_j \cos \lambda_j (t - t_j)$$

of

(5) $$x(t) = \sum_{j=1}^{\infty} r_j \cos \lambda_j (t - t_j); \qquad \sum_{j=1}^{\infty} r_j = R < +\infty.$$

Then ¶

(6) $$\rho_{k+1}(\xi) = \int_{-\infty}^{+\infty} \rho_k(\xi - \eta)\, d\sigma_{k+1}(\eta); \qquad \rho_1(\xi) = \sigma_1(\xi),$$

where $\sigma_j(\xi)$ denotes the distribution function belonging to the periodic function

(7) $$a_j(t) = r_j \cos \lambda_j (t - t_j);$$

i. e.

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* *Loc. cit. I.*

† H. Bohr, "Kleinere Beiträge zur Theorie der fastperiodischen Funktionen. II.," *Det Kgl. Danske Videnskabernes Selskab. Meddelelser*, Vol. 10, No. 10 (1930).

‡ For let the continuous function $x(t)$ be periodic with the period 1 and let it be of bounded variation in the fundamental region $0 \leqq t \leqq 1$. Suppose further that $x(t)$ is zero when $|t - n| \leqq 1/4$, $n = 0, \pm 1, \pm 2, \cdots$ but that $x(t) \neq 0$ for all other values of t. Since the Fourier partial sum $x_k(t)$ is a periodic trigonometric polynomial, its distribution function $\rho_k(\xi)$ is everywhere continuous. Furthermore, $x_k(t)$ approaches the limit $x(t)$ uniformly when $k \to \infty$. Finally, the limit (3) exists for every ξ inasmuch as $x(t)$ is periodic. The limit $\rho(\xi)$ of $\rho_k(\xi)$ possesses, however, a discontinuity at $\xi = 0$.

§ A. Wintner, "Ueber die Stetigkeit der asymptotischen Verteilungsfunktion bei inkommensurablen Partialschwingungen," *Mathematische Zeitschrift*, Vol. 37 (1933), not yet appeared.

¶ *Loc. cit. II.* The recursion formula (6) yields a k-fold iterated Stieltjes integral for $\rho_{k+1}(\xi)$, viz.

$$\rho_{k+1}(\xi) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \sigma_1(\xi - \eta_1)\, d\sigma_2(\eta_1 - \eta_2) \cdots d\sigma_k(\eta_{k-1} - \eta_k)\, d\sigma_{k+1}(\eta_k).$$

This detailed representation takes the place of the shortened expression (18) in the paper *II*, a formula whose meaning is obvious from (19), *loc. cit. II.*

$$(8)\qquad \begin{cases} \sigma_j(\xi) = 0 \quad \text{for} \quad -\infty < \xi < -r_j \\ \sigma_j(\xi) = 1 - [\arccos(\xi/r_j)]/\pi \quad \text{for} \quad -r_j \leqq \xi \leqq r_j \\ \sigma_j(\xi) = 1 \quad \text{for} \quad r_j < \xi < +\infty \end{cases}$$

where $0 \leqq \arccos \leqq \pi$. Furthermore,

$$(9)\qquad \rho(\xi) = \lim_{k=\infty} \rho_k(\xi)$$

holds for those * values of ξ which are continuity points of $\rho(\xi)$; hence (9) holds for all values of ξ. Finally,†

$$(10)\qquad L(s;\, \rho_k) = \prod_{j=1}^{k} L(s;\, \sigma_j)$$

where $L(s;\, \nu)$ denotes the Laplace-Fourier transform

$$(11)\qquad L(s;\, \nu) = \int_{-\infty}^{+\infty} \exp(is\xi)\, d\nu(\xi)$$

of the typical distribution function $\nu(\xi)$ and s is an arbitrary real or complex parameter. Since $|x_k(t)|$ and $|x(t)|$ are, according to (4) and (2), not larger than R, it follows from the definition (3) of a distribution function that

$$\rho_k(\xi) = 0 \quad \text{for} \quad -\infty < \xi < -R, \qquad \rho_k(\xi) = 1 \quad \text{for} \quad R < \xi < +\infty$$

and

$$(12)\qquad \rho(\xi) = 0 \quad \text{for} \quad -\infty < \xi < -R, \qquad \rho(\xi) = 1 \quad \text{for} \quad R < \xi < +\infty.$$

Accordingly, all Stieltjes integrations $\int_{-\infty}^{+\infty}$ may be replaced by $\int_{-R}^{R}$. Hence, from (9) and (11),

$$(13)\qquad \lim_{k=\infty} L(s;\, \rho_k) = L(s;\, \rho)$$

by virtue of the Helly theorem on term-by-term integration.‡ On comparing (10) with (13) there results the multiplicative relation §

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* *Loc. cit. II.*

† *Loc. cit. II.* Cf. G. Doetsch, "Die Integrodifferentialgleichungen vom Faltungstypus," *Mathematische Annalen*, Vol. 89 (1923), pp. 192-207.

‡ E. Helly, "Ueber lineare Funktionaloperationen," *Sitzungsberichte der mathematisch-naturwissenschaftlichen Klasse der Kaiserl. Akademie der Wissenschaften zu Wien*, Vol. 121 (1912), pp. 265-297.

§ The existence of the infinite product (14) is for all values of s assured by (13). Since $J_0(0) = 1$, there follows from (15) and (2) by Schwarz's Lemma a finer result, viz. the uniform convergence of the series

$$\sum_{j=1}^{\infty} |L(s;\, \sigma_j) - 1|$$

in every fixed s-circle. Similar remarks hold regarding the infinite products occurring later on.

$$(14) \qquad L(s;\ \rho) = \prod_{j=1}^{\infty} L(s;\ \sigma_j)$$

expressing the statistical independence * of the distributions σ_j belonging to the partial vibrations (7) of (5).

From (11) and (8) we have

$$L(s;\ \sigma_j) = (1/\pi) \int_{-r_j}^{r_j} (r_j^2 - \xi^2)^{-\frac{1}{2}} \exp(is\xi)\, d\xi,$$

i. e.

$$L(s;\ \sigma_j) = (2/\pi) \int_0^1 (1 - \xi^2)^{-\frac{1}{2}} \cos(sr_j\xi)\, d\xi,$$

or, on placing $\xi = \cos\theta$,

$$L(s;\ \sigma_j) = (2/\pi) \int_0^{\pi/2} \cos(sr_j \cos\theta)\, d\theta.$$

Hence †

$$(15) \qquad L(s;\ \sigma_j) = J_0(r_j s).$$

From (11), (14) and (15) there results

$$(16) \qquad L(s;\ \rho) = \int_{-\infty}^{+\infty} \exp(is\xi)\, d\rho(\xi) = \prod_{j=1}^{\infty} J_0(r_j s).$$

We notice here that the distribution of $x(t)$ is symmetric with respect to the origin, i. e.

$$(17) \qquad \rho(\xi) + \rho(-\xi) = 1.$$

On account of (9) it is sufficient to prove that

$$(18) \qquad \rho_k(\xi) + \rho_k(-\xi) = 1$$

holds for every k. Now from (8)

$$(19) \qquad \sigma_j(\xi) + \sigma_j(-\xi) = 1.$$

Hence (18) holds for $k = 1$ inasmuch as $\rho_1 = \sigma_1$. Suppose that (18) holds for a fixed value of k. Since from (6)

$$\rho_{k+1}(-\xi) = \int_{-\infty}^{+\infty} \rho_k(-\xi - \eta)\, d\sigma_{k+1}(\eta) = -\int_{-\infty}^{+\infty} \rho_k(-\xi + \zeta)\, d\sigma_{k+1}(-\zeta),$$

where $\eta = -\zeta$, there results from (18) and (19) the equality

* Cf. F. Hausdorff, "Beitraege zur Wahrscheinlichkeitsrechnung," *Berichte über die Verhandlungen der Königl. Saechsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-physikalische Klasse,* Vol. 53 (1901), pp. 152-178. This paper discusses also the general methods in Calculus of Probability, which have a connection with the present problem.

† R. Courant und D. Hilbert, *Methoden der mathematischen Physik, I.,* 1924, p. 393.

$$\rho_{k+1}(-\xi) = -\int_{-\infty}^{+\infty}[1-\rho_k(\xi-\zeta)]\, d[1-\sigma_{k+1}(\zeta)],$$

i. e.

$$\rho_{k+1}(-\xi) = \int_{-\infty}^{+\infty} d\sigma_{k+1}(\zeta) - \int_{-\infty}^{+\infty}\rho_k(\xi-\zeta)\, d\sigma_{k+1}(\zeta),$$

or, by virtue of (6),

$$\rho_{k+1}(\xi) + \rho_{k+1}(-\xi) = \int_{-\infty}^{+\infty} d\sigma_{k+1}(\zeta) = 1,$$

inasmuch as the last integral represents the total variation of a monotone function (8). Hence (18) holds for every k. From (16) and (17) there results

$$(20) \qquad L(s;\, \rho) = 2\int_0^{+\infty}\cos(s\eta)\, d\rho(\eta) = \prod_{j=1}^{\infty} J_0(r_j s);$$

hence, by virtue of (12),

$$(21) \qquad 2\int_0^{+\infty}\sin(s\eta)/\eta\, d\rho(\eta) = \int_0^s \prod_{j=1}^{\infty} J_0(r_j l)\, dl.$$

For positive values of the independent variable we need the appraisals

$$(22) \qquad \left| \prod_{j=1}^{\infty} J_0(r_j\eta) \right| < \Gamma_m/\eta^m; \qquad (m = 0, 1, 2, \cdot\ \cdot),$$

where Γ_m is a constant depending upon m but independent of $\eta > 0$. First, the well-known asymptotic formula *

$$J_0(\eta) \sim \eta^{-\frac{1}{2}}(2/\pi)^{\frac{1}{2}}\cos(\eta - \pi/4); \qquad \eta \to +\infty$$

assures the existence of a constant C for which

$$| J_0(\eta) | < C/\eta^{\frac{1}{2}}.$$

Accordingly,

$$\left| \prod_{j=1}^{2m} J_0(r_j\eta) \right| < \Gamma_m/\eta^m,$$

where $\Gamma_m = C^{2m}/(r_1 r_2 \cdot \cdot \cdot r_{2m-1} r_{2m})^{\frac{1}{2}}$. Hence (22) is obvious inasmuch as

$$J_0(X) = \int_0^{2\pi}\cos(X\cos\theta)\, d\theta/2\pi$$

has, for real values of X, a modulus $\leqq 1$.

We now restrict s in (21) to real and non-negative values and write ξ instead of s. Thus

$$2\int_0^{+\infty}\sin(\xi\eta)/\eta\, d\rho(\eta) = \int_0^{\xi}\prod_{j=1}^{\infty} J_0(r_j\eta)\, d\eta; \qquad \xi \geqq 0.$$

* Courant-Hilbert, *op. cit.*, p. 435.

This integral equation for the monotone continuous function ρ may be solved, by virtue of (12), by means of the Gauss-Fourier inversion formula which yields *

$$(23) \qquad \rho(\xi) - \rho(0) = (1/\pi) \int_0^{+\infty} \{[\sin(\xi\eta)/\eta] \prod_{j=1}^{\infty} J_0(r_j\eta)\} \, d\eta$$

for $\xi \geqq 0$. It is clear from (17) that (23) holds for $\xi < 0$ also. The expression

$$(1/\pi) \int_0^{+\infty} (d^k\{\cdot \cdot \cdot\}/d\xi^k) \, d\eta$$

resulting from (23) by k-fold formal differentiation is, by virtue of (22), absolutely and uniformly convergent for $-\infty < \xi < +\infty$. In order to see this, it is sufficient to choose $m \geqq k + 2$. Since m, and therefore k, may be chosen arbitrarily large, it follows † that *the distribution function* $\rho(\xi)$ *possesses for* $-\infty < \xi < +\infty$ *derivatives of arbitrarily high order.*

Hence from (17)

$$(24) \qquad \rho(0) = \tfrac{1}{2}, \qquad \rho^{(k)}(0) = 0; \qquad (k = 2, 4, 6, \cdots).$$

Similarly from (12)

$$(25) \qquad \rho^{(k)}(R) = 0, \qquad \rho^{(k)}(-R) = 0; \qquad (k = 1, 2, 3, \cdots),$$

although $\rho(\xi)$ is known ‡ to be nowhere constant in the range $-R \leqq \xi \leqq R$. Thus the behavior of $\rho(\xi)$ at $\xi = \pm R$ is the same as that of Cauchy's example

$$\exp(-1/\xi^2)$$

at $\xi = 0$.

Let us notice that the distribution function $\rho_k(\xi)$ belonging to the finite sum (4) cannot possess derivatives of arbitrarily high order if k has a fixed value. Correspondingly, infinitely many appraisals (22) break down if the infinite product is replaced by a finite one.

First, $\rho_1 = \sigma_1$ is everywhere continuous, its derivative is, however, infinite at $\xi = \pm r_1$. The function ρ_2 has been considered by Bessel in his celebrated

* The validity of the Gauss-Fourier inversion formula (cf. F. Hausdorff, *loc. cit.*), which is at present (23), is assured under conditions which are essentially more general than (12). Cf., for instance, T. C. Burkill, "The expression in Stieltjes integrals of the inversion formulae of Fourier and Hankel," *Proceedings of the London Mathematical Society*, Series 2, Vol. 25 (1926), pp. 513-524.

† Cf., for instance, E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, Second Edition, Vol. II, p. 359, Cambridge University Press, 1926.

‡ *Loc. cit. I.* It follows that the function $x(t)$ takes on every value between $-R$ and R. The latter fact is contained in the Kronecker approximation theorem also.

paper on the Gaussian frequency curve.* The first derivative of ρ_2 is a complete elliptic integral of the first kind and is infinite at the four points $\xi = \pm (r_1 + r_2)$, $\xi = \pm (r_1 - r_2)$, two of which coincide when $r_1 = r_2$. The function ρ_3 possesses everywhere a continuous first derivative but the second derivative is infinite at some points, and so on, so that ρ_k is the smoother, the farther we go in Bessel's statistical † iteration process (6).

It is clear from (2) that the limit function ρ cannot be related to the Gaussian frequency curve.‡

The Markoff condition for the validity of the Gauss law § takes in our case the form

$$\lim_{k=\infty} S_{2n}(k) : S_2(k) = 0, \ (n = 2, 3, \cdots), \text{ where } S_n(k) = (\sum_{j=1}^{k} r_j^n)^{1/n}.$$

This condition is, however, not a necessary one (Liapounoff).

An Integral Equation for the Central Waves.

For later purposes (cf. p. 327) we consider in the present chapter a function $\delta(r)$ implicitly defined for $0 \leqq r \leqq R$ as a continuous solution of the functional equation

$$(26) \qquad \rho'(r) = 2\int_r^R (q^2 - r^2)^{-\frac{1}{2}} q\delta(q)\,dq; \qquad 0 \leqq r \leqq R.$$

There exists exactly one such function and it possesses, save at the origin $r = 0$, derivatives of arbitrarily high order. Furthermore,

$$(27) \qquad \delta^{(k)}(R) = 0, \qquad (k = 0, 1, 2, \cdots).$$

Finally,

$$(28) \qquad \int_0^{2\pi}\int_0^R r\delta(r)\exp\{i(u\cos\vartheta + v\sin\vartheta)r\}dr\,d\vartheta = L(\{u^2 + v^2\}^{\frac{1}{2}};\ \rho)$$

where u and v are arbitrary real or complex parameters.

In order to prove these statements we first reduce (26) to Abel's integral equation

* F. W. Bessel, *Abhandlungen*, Vol. 2 (1876), pp. 378-380.

† Cf. also H. Bohr and B. Jessen, "Om Sandsynlighedsfordelinger ved Addition af konvekse Kurven," *Det Kgl. Danske Videnskabernes Selskabs Skrifter*, Series 8, Vol. 12 (1929), No. 3.

‡ Cf. in this connection F. Hausdorff, *loc. cit.*

§ Cf. R. Deltheil, *Erreurs et moindres carrés*, Paris, 1930, pp. 71-74; M. Fréchet and J. Shohat, "A proof of the generalized central limit theorem in the theory of probability," *Transactions of the Mathematical Society*, Vol. 33 (1931), pp. 533-543.

(29) $$\chi(X) = \int_0^X (X-Y)^{-\frac{1}{2}} \tau(Y)\,dY; \qquad 0 \leqq X \leqq R^2$$

by placing

(30) $$X = R^2 - r^2, \qquad Y = R^2 - q^2$$

and

(31) $$\chi(X) = \rho'(\sqrt{R^2 - X}), \qquad \tau(X) = \delta(\sqrt{R^2 - X})$$

(hence χ is given and τ is the unknown function). Since $\rho(\xi)$ has for every ξ derivatives of any order, the function $\chi(X)$ possesses, according to (31), derivatives of arbitrarily high order in the half-open range $0 \leqq X < R^2$; furthermore, by virtue of (24), (25) and (31),

(32) $$\chi^{(k)}(0) = 0, \qquad (k = 0, 1, 2, \cdots),$$

and the first derivative $\chi'(X)$ exists and is continuous in the closed range $0 \leqq X \leqq R^2$. Hence * (29) has exactly one continuous solution τ in this closed range, viz. the one represented by Abel's inversion formula

(33) $$\tau(X) = \int_0^X (X-Y)^{-\frac{1}{2}} \chi'(Y)\,dY/\pi; \qquad 0 \leqq X \leqq R^2.$$

On combining (30) and (31) with (33) we see that (26) possesses the unique continuous solution

(34) $$\delta(r) = -\int_r^R (q^2 - r^2)^{-\frac{1}{2}} \rho''(q)\,dq/\pi; \qquad 0 \leqq r \leqq R.$$

We have now to prove that in the half-open range $r \neq 0$ all derivatives of $\delta(r)$ exist and satisfy the relations (27). In other words [cf. (30), (31)], we have to prove that in the half-open range $0 \leqq X < R^2$ the function $\tau(X)$ possesses derivatives of arbitrarily high order which all vanish for $X = 0$.

Since X is supposed to be $\neq R^2$, we know that $\chi^{(k)}(X)$ exists for every k and for all values of X under consideration. Hence, from (32),

(35) $$(X-Y)^{\frac{1}{2}} \chi^{(k)}(Y) = 0 \text{ for both } Y = 0 \text{ and } Y = X.$$

On writing (33) in the form

(36) $$\tau(X) = -2\int_0^X \{d(X-Y)^{\frac{1}{2}}/dY\} \chi'(Y)\,dY/\pi; \qquad 0 \leqq X < R^2$$

and applying partial integration, the boundary condition (35) yields

(37) $$\tau(X) = 2\int_0^X (X-Y)^{\frac{1}{2}} \chi''(Y)\,dY/\pi; \qquad 0 \leqq X < R^2.$$

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* Cf. the definitive results of L. Tonelli, "Su un problema di Abel," *Mathematische Annalen*, Vol. 99 (1928), pp. 185-192.

Hence $\tau'(X)$ exists, viz.

$$(38)\qquad \tau'(X) = \int_0^X (X-Y)^{-1/2}\chi''(Y)\,dY/\pi; \qquad 0 \leqq X < R^2.$$

Since all derivatives of $\chi(Y)$ exist for $0 \leqq Y \leqq X$ and (35) holds for every k, the process which led from (33) to (38) may be repeated indefinitely, i. e. all derivatives $\tau^{(k)}(X)$ exist and

$$(39)\qquad \tau^{(k)}(X) = \int_0^X (X-Y)^{-1/2}\chi^{(k+1)}(Y)\,dY/\pi; \qquad 0 \leqq X < R^2.$$

Finally from (39)

$$(40)\qquad \tau^{(k)}(0) = 0.$$

Q. E. D.

We now prove (28). The even momentum

$$\int_0^R r^{2n}\rho'(r)\,dr$$

of ρ' is, according to (26),

$$= 2\int_0^R r^{2n}\left[\int_r^R (q^2-r^2)^{-1/2} q\delta(q)\,dq\right] dr,$$

i. e., by Dirichlet's rule,*

$$= \int_0^R \left[2\int_0^q r^{2n}(q^2-r^2)^{-1/2}\,dr\right] q\delta(q)\,dq$$

or (on placing $r = qp$ where q is fixed)

$$= \int_0^R \left[2\int_0^1 (qp)^{2n}(1-p^2)^{-1/2}\,dp\right] q\delta(q)\,dq.$$

Hence

$$\int_0^R r^{2n}\rho'(r)\,dr = \left[\int_0^R q^{2n} q\delta(q)\,dq\right]\left[2\int_0^1 p^{2n}(1-p^2)^{-1/2}\,dp\right],$$

where †

$$\left[2\int_0^1 p^{2n}(1-p^2)^{-1/2}\,dp\right] = 2\int_0^{\pi/2}\cos^{2n}\theta\,d\theta = \pi(2n)!/(n!^2\, 2^{2n}).$$

Accordingly,

$$\pi\int_0^R r^{2n+1}\delta(r)\,dr/(n!^2\, 2^{2n}) = \int_0^R r^{2n}\rho'(r)\,dr/(2n)!$$

or

$$\pi\int_0^R (-s^2r^2/4)^n\, r\delta(r)\,dr/(n!^2) = \int_0^R (-s^2r^2)^n\rho'(r)\,dr/(2n)!,$$

* Cf., for instance, L. Tonelli, *loc. cit.*

† Cf. in this connection G. Pólya, "Application of a theorem connected with the problem of moments," *The Messenger of Mathematics*, Vol. 55 (1926), pp. 189-192.

where s is arbitrary. This may be written, by virtue of the developments

$$J_0(sr) = \sum_{n=0}^{\infty} (-s^2r^2/4)^n/(n!^2), \qquad \cos(sr) = \sum_{n=0}^{\infty} (-s^2r^2)^n/(2n)!,$$

in the form

$$\pi \int_0^R J_0(sr) r\delta(r)\, dr = \int_0^R \cos(sr)\rho'(r)\, dr,$$

the legality of the term-by-term integration being trivial. Hence from (20)

$$(41) \qquad 2\pi \int_0^R J_0(sr) r\delta(r)\, dr = L(s;\ \rho).$$

On the other hand,*

$$\int_0^{2\pi} \exp\{i \cos(w\vartheta)\} d\vartheta = 2\pi J_0(w),$$

i. e.

$$(42) \quad \int_0^{2\pi} \exp\{i(u\cos\vartheta + v\sin\vartheta)r\} d\vartheta = 2\pi J_0(rs) \text{ where } s = \{u^2 + v^2\}^{1/2}.$$

On substituting (42) in (41) there results (28).

The continuous function $\delta(r)$ has so far been defined for $0 \leqq r \leqq R$ only. It will be convenient to set

$$(43) \qquad \delta(r) = 0 \quad \text{for} \quad R < r < +\infty.$$

By virtue of (27) this extended function δ possesses derivatives of any order for $0 < r < +\infty$.

The Laplace Transform of the Time Averages.

It is supposed that the frequencies λ_j of (1) are linearly independent. Hence if n, m, k denote arbitrary non-negative integers,

$$(44) \quad \lim_{T=+\infty} (1/2T) \int_{-T}^{T} [\sum_{j=1}^{k} r_j \cos\lambda_j(t-t_j)]^n [\sum_{j=1}^{k} r_j \sin\lambda_j(t-t_j)]^m dt$$
$$= (1/2\pi)^k \int_0^{2\pi} \cdots \int_0^{2\pi} [\sum_{j=1}^{k} r_j\cos\theta_j]^n [\sum_{j=1}^{k} r_j \sin\theta_j]^m d\theta_1 \cdots d\theta_k,$$

where $\theta_1, \cdots, \theta_k$ are k independent integration variables. This well-known identity may be verified either by complete induction or else directly and yields, according to Bohr, a simple proof for the Kronecker approximation theorem. We shall use (44) as in the paper *II* for purposes which are finer ‡

* Cf. Courant-Hilbert, *op. cit.*, p. 390.

† Cf. E. C. Titchmarsh, *The zeta-function of Riemann*, Cambridge University Press, 1930, p. 98.

‡ Cf. the introduction of the paper *II*, referred to on p. 310.

than the Kronecker theorem. In fact, we shall extend the *statistical* relation (14) to the case of the complex-valued distribution (1).

Let $f(t) = g(t) + ih(t)$ be an almost-periodic (hence * continuous and bounded) function of the real variable t, where g and h are real, and let $\{f_k(t)\}$ denote a sequence of such functions. The exponential

$$\exp i\{ug(t) + vh(t)\} \quad \text{where} \quad f = g + ih \tag{45}$$

is an almost-periodic function of t for all real and complex values of the parameters u, v, inasmuch as (45) is a uniform limit † of such functions; in fact,

$$\| g^n h^m \| \leqq \| g + ih \|^{n+m}, \tag{46}$$

where $\| q \|$ denotes the least upper bound of $| q(t) |$ in the infinite range $-\infty < t < +\infty$. Obviously

$$\lim_{k=\infty} \| \exp i\{ug_k + vh_k\} - \exp i\{ug + vh\} \| = 0 \quad \text{whenever} \quad \lim_{k=\infty} \| f_k - f \| = 0, \tag{47}$$

where $f_k = g_k + ih_k$ and $f = g + ih$. The operator

$$\mathfrak{M}(f) = \lim_{T=+\infty} \int_{-T}^{T} f(t)\, dt/2T \tag{48}$$

is defined ‡ for every almost-periodic function f, hence for the function (45). For the time-average of this exponential we introduce the abbreviation

$$\mathfrak{L}(u, v;\, f) = \mathfrak{M}(\exp i\{ug + vh\}) \quad \text{where} \quad f = g + ih, \tag{49}$$

so that $\mathfrak{L}$ may be considered as the Laplace-Fourier transform of the time-function $f(t)$. Clearly

$$\lim_{k=\infty} \mathfrak{M}(f_k) = \mathfrak{M}(f) \quad \text{whenever} \quad \lim_{k=\infty} \| f_k - f \| = 0. \tag{50}$$

Also, for all values of the parameters u, v,

$$\mathfrak{L}(u, v;\, f) = \sum_{p=0}^{\infty} p\,!^{-1} \sum_{q=0}^{p} C_{pq} (iu)^{p-q} (iv)^q\, \mathfrak{M}(g^{p-q} h^q), \tag{52}$$

where $f = g + ih$ and

$$C_{pq} = p\,!\, (p - q)\,!^{-1}\, q\,!^{-1}. \tag{53}$$

The development (52), resulting formally from (49), is legalized by (50) and (46); in fact, $g(t)^n\, h(t)^m$ is § an almost-periodic function as $f(t) = g(t) + ih(t)$ is.

* H. Bohr, "Fastperiodische Funktionen," *Ergebnisse der Mathematik und ihre Grenzgebiete*, Vol. 1, No. 5 (1932), pp. 29-30.

† H. Bohr, *ibid.*, pp. 31-33.

‡ H. Bohr, *ibid.*, pp. 34-36.

§ H. Bohr, *ibid.*, p. 33.

Let

(55) $$z_k(t) = x_k(t) + iy_k(t) = \sum_{j=1}^{k} c_j(t)$$

denote a partial sum of (1), where

(56) $$r_j \exp i\lambda_j(t - t_j) = r_j \cos \lambda_j(t - t_j) + ir_j \sin \lambda_j(t - t_j) = c_j.$$

Then $\lim_{k=\infty} \| z_k - z \| = 0$ is assured by (2). Hence from (47), (50), (49)

(57) $$\mathfrak{L}(u, v;\, z) = \lim_{k=\infty} \mathfrak{L}(u, v;\, z_k).$$

Furthermore,

(58) $$\mathfrak{L}(u, v;\, c_j) = (1/2\pi) \int_0^{2\pi} \exp i\{(u \cos \theta + v \sin \theta) r_j\}\, d\theta.$$

In fact, (58) holds by virtue of (56) and (52) if and only if

(59) $$\mathfrak{M}([r_j \cos \lambda_j(t - t_j)]^{p-q} [r_j \sin \lambda_j(t - t_j)]^q) = (1/2\pi) \int_0^{2\pi} [r_j \cos \theta]^{p-q} [r_j \sin \theta]^q \, d\theta \qquad (p \geqq q \geqq 0),$$

where we developed the integral $\int_0^{2\pi}$ occurring in (58) according to the powers of u and v. Since j has in (59) a fixed value it is sufficient to prove (59) for $j = 1$, and on placing in (44)

$$n = p - q, \quad m = q, \quad k = 1,$$

there results (59) for $j = 1$. Hence (58) holds true. Also, from (44), (55) and (56),

(60) $$\mathfrak{M}(x_k^{p-q} y_k^q) = (1/2\pi)^k \int_0^{2\pi} \cdots \int_0^{2\pi} [\sum_{j=1}^{k} r_j \cos \theta_j]^{p-q} [\sum_{j=1}^{k} r_j \sin \theta_j]^q \, d\theta_1 \cdots d\theta_k$$

where $p \geqq q \geqq 0$.

On replacing in (52) the typical function $f(t) = g(t) + ih(t)$ by the function (55), it follows from (60) that

$$\mathfrak{L}(u, v;\, z_k) = \sum_{p=0}^{\infty} p!^{-1} \sum_{q=0}^{p} C_{pq} (iu)^{p-q} (iv)^q (1/2\pi)^k \times \int_0^{2\pi} \cdots \int_0^{2\pi} [\sum_{j=1}^{k} r_j \cos \theta_j]^{p-q} [\sum_{j=1}^{k} r_j \sin \theta_j]^q \, d\theta_1 \cdots d\theta_k.$$

Accordingly from (53)

$$\mathfrak{L}(u, v;\, z_k) = (1/2\pi)^k \sum_{p=0}^{\infty} p!^{-1} \times \int_0^{2\pi} \cdots \int_0^{2\pi} \{iu \sum_{j=1}^{k} r_j \cos \theta_j + iv \sum_{j=1}^{k} r_j \sin \theta_j\}^p \, d\theta_1 \cdots d\theta_k,$$

which may be written in the form

$$\mathfrak{L}(u, v;\ z_k) = (1/2\pi)^k \times \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{p=0}^{\infty} p\,!^{-1} \{ \sum_{j=1}^{k} (iur_j \cos\theta_j + ivr_j \sin\theta_j) \}^p \, d\theta_1 \cdots d\theta_k,$$

the legality of the term-by-term integration being trivial. Consequently,

$$\mathfrak{L}(u, v;\ z_k) = (1/2\pi)^k \int_0^{2\pi} \cdots \int_0^{2\pi} \exp \sum_{j=1}^{k} (iur_j \cos\theta_j + ivr_j \sin\theta_j)\, d\theta_1 \cdots d\theta_k,$$

or

$$= (1/2\pi)^k \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j=1}^{k} \exp(iur_j \cos\theta_j + ivr_j \sin\theta_j)\, d\theta_j,$$

i. e.

$$\mathfrak{L}(u, v;\ z_k) = \prod_{j=1}^{k} (1/2\pi) \int_0^{2\pi} \exp ir_j\{u \cos\theta_j + v \sin\theta_j\} d\theta_j.$$

Hence from (58) and (57)

$$\mathfrak{L}(u, v;\ z) = \prod_{j=1}^{\infty} \mathfrak{L}(u, v;\ c_j). \tag{61}$$

The multiplicative rule (61) is analogous to (14). The expressions $\mathfrak{L}$ are, however, time-averages whereas the integrals L represent space-integrals extended over the one-dimensional phase-space.* We shall now transform the time-averages $\mathfrak{L}$ in space-integrals Λ extended over the present phase-space which is the plane (x, y).

The Statistical Independence.

Let R denote the least upper bound of $|z(t)|$, where $z(t) = x(t) + iy(t)$ is an almost-periodic function. We do not suppose, at present, that the frequencies λ_j are linearly independent. Let Q be a rectangle in the (x, y)-plane parallel to the coördinate axes, and let $\{Q;\ T\}$ denote the set of those values t in the interval $|t| < T$ for which the point $x = x(t)$, $y = y(t)$ is within Q. In a recent paper Haviland † proves the following theorems:

(I). Every almost-periodic function $z(t)$ does possess a distribution function. In a more precise manner, there exists a monotone ‡ absolutely additive § set-function $\phi(E)$ such that

* Cf. in this connection G. D. Birkhoff, "Proof of the Ergodic Theorem," *Proceedings of the National Academy of Sciences*, Vol. 17 (1931), pp. 650-660.

† E. K. Haviland, *loc. cit.* The order of presentation of these theorems differs in his paper from that given here.

‡ J. Radon, "Theorie und Anwendungen der absolut additiven Mengenfunktionen," *Sitzungsberichte der mathematisch-naturwissenschaftlichen Klasse der Kaiserl. Akademie der Wissenschaften zu Wien*, Vol. 122 (1913), pp. 1295-1438 (more particularly p. 1303) and "Ueber lineare Funktionaltransformationen und Funktionalgleichungen," *ibid.*, Vol. 128 (1919), pp. 1083-1121.

§ J. Radon, *loc. cit.*, p. 1299.

$$\lim_{T=+\infty} \text{meas}\,\{Q;\ T\}/2T \text{ exists and } = \phi(Q),$$

provided that none of the four boundary lines of Q lies on a certain denumerable set of lines $x = x_j$, $y = y_k$. These are termed singular lines of ϕ.

(II). These lines cannot exist if * the total variation of $\phi(E)$ in Q is an absolutely continuous set-function of Q. On the other hand, there exist † almost-periodic functions $z(t)$ having actually a singular line $x = x_j$ or $y = y_k$.

(III). Since $|z(t)| \leqq R$ for every t, it is clear from (I) that $\phi(E)$ vanishes for all rectangles $E = Q$ without the circle $x^2 + y^2 \leqq R^2$. Hence ‡ the double Stieltjes integral

$$\int\int_{-\infty}^{+\infty} P(x, y)\, d\phi(E)$$

exists for every continuous point-function $P(x, y)$. In particular, all momenta

$$\int\int x^n y^m d\phi(E); \qquad (n, m = 0, 1, 2, \cdots)$$

of ϕ exist. Here and always if not otherwise indicated the integration is extended over any region containing the circle $x^2 + y^2 \leqq R^2$, e. g. over the whole (x, y)-plane.

(IV). The momenta of $\phi(E)$ are the corresponding time-momenta of $z(t) = x(t) + iy(t)$:

$$\int\int x^n y^m\, d\phi(E) = \lim_{T=+\infty} (1/2T) \int_{-T}^{T} x(t)^n\, y(t)^m\, dt,$$

where $n, m = 0, 1, 2, \cdots$.

(V). If an absolutely additive set-function $\omega(E)$ vanishes § for all rectangles without a sufficiently large circle, and if the momenta of $\omega(E)$ represent the corresponding time-momenta of $z(t)$, then ω is identical ¶ with the distribution function ϕ of $z(t)$ although it is not *presupposed* that ω be monotone.

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* Cf. J. Radon, *loc. cit.*, pp. 1320-1322 and pp. 1093-1094.

† Cf. Bohr's example referred to above (p. 311).

‡ Cf. J. Radon, *loc. cit.*, pp. 1322-1324.

§ It may be shown that this restriction can be omitted. We do not need, however, this extension of the uniqueness theorem.

¶ This is to mean that $\omega(Q) = \phi(Q)$ holds for all those rectangles Q which are not excluded by (I). The actual value of the monotone set-function ϕ for the "singular" rectangles is undetermined and immaterial in the same sense as is the actual value of a monotone function $\rho(\xi)$ at a discontinuity point $\xi = \xi_m$. Cf. the papers of Radon and Haviland, referred to above.

These theorems of Haviland correspond to those results regarding a real-valued almost-periodic function which are proven in my first paper, referred to on p. 309, footnote †. We know that the latter results may essentially be refined if the frequencies λ_j be linearly independent. In this case we found explicit results instead of the mere existence theorem (3). We shall now extend these explicit results to complex-valued almost-periodic functions with linearly independent frequencies. This case is of first importance in the analytic theory of numbers. Even without the assumption of linear independence we have as a consequence of Haviland's results the following

LEMMA. *Let $\phi(E)$ denote the distribution function of the almost-periodic function $z(t)$. Set*

$$\Lambda(u, v;\ \omega) = \int\int \exp i\{ux + vy\}\, d\omega(E) \tag{62}$$

where $\omega(E)$ is any absolutely additive set-function vanishing without a sufficiently large circle $x^2 + y^2 \leqq R^2$. Then *

$$\omega = \phi \tag{63a}$$

holds if and only if

$$\Lambda(u, v;\ \omega) = \mathfrak{L}(u, v;\ z) \tag{63b}$$

for all values of the arbitrary parameters u and v.

In fact, on placing

$$\mathrm{M}_{nm}(\omega) = \int\int x^n y^m d\omega(E),$$

we have from (62) and (53)

$$\Lambda(u, v;\ \omega) = \sum_{p=0}^{\infty} p\,!^{-1} \sum_{q=0}^{p} C_{pq}(iu)^{p-q}(iv)^q\, \mathrm{M}_{p-q\ q}(\omega),$$

the legality of the term-by-term integration being trivial. On the other hand, from (52),

$$\mathfrak{L}(u, v;\ z) = \sum_{p=0}^{\infty} p\,!^{-1} \sum_{q=0}^{p} C_{pq}(iu)^{p-q}(iv)^q\, \mathfrak{M}(x^{p-q}y^q),$$

where $z(t) = x(t) + iy(t)$. On comparing the coefficients of these integral power series we see that (63b) is equivalent to

$$\mathrm{M}_{nm}(\omega) = \mathfrak{M}(x^n y^m);\qquad (n, m = 0, 1, 2, \cdots). \tag{63c}$$

Now (63c) follows from (63a) by (IV), and (63a) follows from (63c) by (V), so that (63a) is equivalent to (63c). Hence (63a) is equivalent to (63b).

* Cf. the previous footnote.

According to the Lemma thus proven we have

$$\mathfrak{L}(u, v;\ z) = \Lambda(u, v;\ \phi) \tag{64}$$

and

$$\mathfrak{L}(u, v;\ c_j) = \Lambda(u, v;\ \psi_j), \tag{65}$$

where ψ_j denotes the distribution function of the periodic function $c_j(t) = r_j \exp i\lambda_j(t - t_j)$. On substituting (64) and (65) in (61) we obtain the statistical independence relation

$$\Lambda(u, v;\ \phi) = \prod_{j=1}^{\infty} \Lambda(u, v;\ \psi_j), \tag{66}$$

which is by virtue of (1) and (56) the two-dimensional analogue of (14).

On comparing (66) with (14) and using the Abel integral equation (26) we shall now *calculate* the distribution function of (1) in terms of the one-dimensional distribution function ρ, which we know by the explicit representation (23). It would not be difficult to consider spaces with more than two dimensions. Besides, the treatment of spaces with an odd number of dimensions is simpler insofar as no Abel integral equation occurs. The occurance of this integral equation in the case of an even dimension number is related to well-known facts regarding Huyghens' Principle.*

The Distribution Function.

The total variation of a distribution function $\phi(E)$ belonging to an arbitrary almost-periodic function $z(t)$ is $= 1$. In fact, on placing both exponents n, m in (IV) equal to zero, there results

$$\int\int d\phi(E) = 1. \tag{67}$$

Since $\phi(E)$ is by (I) monotone and $\geqq 0$ we conclude that

$$0 \leqq \phi(E) \leqq 1 \tag{68}$$

for every E.

Let $D(x, y)$ be a continuous point-function which is $= 0$ when $x^2 + y^2 \geqq R^2$. Then

$$\omega(E) = \int\int_E D(x, y)\, dx\, dy \tag{69}$$

* Cf. Philomena Mäder, "Ueber die Darstellung von Punktfunktionen im n-dimensionalen euklidischen Raum durch Ebenenintegrale," *Mathematische Zeitschrift*, Vol. 26 (1927), pp. 646-652. This paper contains also references to previous investigations. Cf. also J. Hadamard, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Paris, 1932, *passim*.

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is an absolutely additive set-function which vanishes for all rectangles without the circle $x^2 + y^2 \leqq R^2$. Furthermore,

$$\int\int P(x,y)\,d\omega(E) = \int\int P(x,y)D(x,y)\,dx\,dy \tag{70}$$

for any continuous point-function $P(x,y)$. For we have from (69)

$$\omega(Q_{ik}) = D(\xi_{ik}, \eta_{ik}) \mid Q_{ik} \mid, \tag{69a}$$

where (ξ_{ik}, η_{ik}) is some point in the interior or on the boundary of the rectangle Q_{ik}, and $\mid Q_{ik} \mid$ denotes the area of Q_{ik}. Accordingly,

$$\sum_i \sum_k P(\xi_{ik}, \eta_{ik})\omega(Q_{ik}) = \sum_i \sum_k P(\xi_{ik}, \eta_{ik}) D(\xi_{ik}, \eta_{ik}) \mid Q_{ik} \mid \tag{70a}$$

for every partition of the square $\mid x \mid \leqq R$, $\mid y \mid \leqq R$ in rectangles Q_{ik}. On considering a sequence of partitions in such a way that the maximum diameter of the rectangles occurring in the n-th partition approaches zero when $\lim n = \infty$, equation (70) follows from (70a) by the integral definitions of Radon and Riemann respectively.

If the distribution function $\phi(E)$ of an almost-periodic function $z(t)$ possesses a representation (69), it is clear from (II) that the sequence of singular lines mentioned under (I) cannot exist, i. e. that

$$\lim_{T=\infty} \text{meas}\,\{Q;\ T\}/2T = \int\int_Q D(x,y)\,dx\,dy \tag{71}$$

holds for *every* Q. If the frequencies of the almost-periodic function $z(t)$ be linearly independent, its distribution function may be represented, according to Jessen, in the form (69), provided that $z(t)$ is analytic by virtue of its representation as an absolutely convergent Dirichlet series (cf. p. 310 above). The distribution function $\psi_j(E)$ belonging to the partial vibration (56) of (1) does not allow a representation (69). More than that, there does not exist a measurable function possessing over E a Lebesgue integral $= \psi_j(E)$. In fact, the very definition of a distribution function yields from (56) the relation

$$2\pi r_j \psi_j(E) = \text{length of the arc } E_j, \tag{72}$$

where E_j denotes that portion of the circle $x^2 + y^2 = r_j{}^2$ which is within the open rectangle E, provided that there exist such a portion; otherwise $\psi_j(E) = 0$. Now this set-function is clearly not absolutely continuous and therefore does not allow a Lebesgue representation.

From (72) and (62) we obtain by the Radon integral definition the formula

$$(73) \qquad \Lambda(u, v;\ \psi_j) = \int_0^{2\pi} \exp\{i(ur_j \cos\theta + vr_j \sin\theta)\}\ d\theta/2\pi$$

where $x = r\cos\theta$, $y = r\sin\theta$. Besides, (73) follows from (58) and (65) also. Now from (73) and (42)

$$(74) \qquad \Lambda(u, v;\ \psi_j) = J_0(r_j\{u^2 + v^2\}^{1/2}).$$

Hence, from (66),

$$(75) \qquad \Lambda(u, v;\ \phi) = \prod_{j=1}^{\infty} J_0(r_j\{u^2 + v^2\}^{1/2}).$$

On comparing (75) with (16) there results

$$(76) \qquad \Lambda(u, v;\ \phi) = L(\{u^2 + v^2\}^{1/2};\ \rho),$$

or, according to (28),

$$(77) \qquad \Lambda(u, v;\ \phi) = \int_0^R \int_0^{2\pi} r\delta(r) \exp\{i(u\cos\vartheta + v\sin\vartheta)r\}\ dr\, d\vartheta.$$

On placing

$$(78) \qquad x = r\cos\vartheta, \qquad y = r\sin\vartheta$$

and applying (70) to the point-function

$$P(x, y) = \exp\{i(ux + vy)\} = \exp\{i(u\cos\vartheta + v\sin\vartheta)r\}$$

and the absolutely additive set-function

$$(79) \quad \omega(E) = \int\int_E \delta(\sqrt{x^2 + y^2})\, dx\, dy;\ \delta(\sqrt{x^2 + y^2}) = 0 \text{ for } x^2 + y^2 \geqq R^2,$$

we see from (62) that

$$(80) \quad \Lambda(u, v;\ \omega) = \int\int \delta(r) \exp\{i(u\cos\vartheta + v\sin\vartheta)r\} dx\, dy;\ x^2 + y^2 = r^2.$$

Since $dx\,dy = r\,dr\,d\vartheta$, it is clear from (43) that the double integrals occurring in (77) and (80) are identical. Consequently

$$\Lambda(u, v;\ \phi) = \Lambda(u, v;\ \omega),$$

or, according to (64),

$$\mathfrak{L}(u, v;\ z) = \Lambda(u, v;\ \omega).$$

Hence from the Lemma

$$(81) \qquad \phi(E) = \omega(E)$$

(p. 324). Since ϕ is monotone by (I) it is clear from (79) that for the distribution function (81) the singular lines not excluded by (I) cannot exist and that (81) holds for *every* rectangle. On comparing (69) with (79) we see that $D(x, y) = \delta(r)$, i. e. that the distribution of (1) is of central symmetry. This is in accordance with the Kronecker-Weyl approximation theorem. Furthermore, from (79),

(82) $$\delta(r) \geqq 0,$$

inasmuch as (81) is monotone by (I).

Accordingly, the asymptotic distribution of the values of every almost-periodic function

$$z(t) = x(t) + iy(t) = \sum_{j=1}^{\infty} r_j \exp i(t - t_j)\lambda_j; \; r_j > 0, \; R = \sum_{j=1}^{\infty} r_j < +\infty$$

with linearly independent frequencies λ_j *possesses a non-negative density of probability* δ *which is a function of* $r^2 = x^2 + y^2$ *alone. This function* $\delta(r)$ *possesses derivatives of arbitrarily high order if* $r \neq 0$ *and remains continuous at the origin* $r = 0$. *The radial density is explicitly given by the formula*

(83a) $$\pi\delta(r) = -\int_r^R (q^2 - r^2)^{-1/2} \rho''(q)\, dq; \; \delta(r) = 0 \text{ for } r \geqq R,$$

where *

(83b) $$\pi\rho''(r) = -\int_0^{+\infty} q \sin(rq) \prod_{j=1}^{\infty} J_0(r_j q)\, dq; \; \rho''(r) = 0 \text{ for } r \geqq R.$$

Also,

(84a) $$\pi\rho^{(2n+1)}(r) = (-1)^n \int_0^{+\infty} q^{2n}\, \Xi(q) \cos(rq)\, dq, \qquad (n = 0, 1, 2, \cdots),$$

and

(84b) $$\pi\rho^{(2n)}(r) = (-1)^n \int_0^{+\infty} q^{2n-1}\, \Xi(q) \sin(rq)\, dq, \qquad (n = 1, 2, \cdots),$$

where †

(85) $$\Xi(q) = \prod_{j=1}^{\infty} J_0(r_j q)$$

and

(86) $$\Xi(q) = O(q^{-m}) \text{ when } q \to +\infty$$

for every fixed value of $m \geqq 0$.

The important point is that the Radon integral notion allows the treatment of "discontinuous" distributions of the type (72). The method is valid also in the case, illustrated by a geometrical investigation by Bohr and

* Cf. p. 316 and p. 315 above.

† The product (85) governs also some other statistical problems. Cf. Lord Rayleigh, "On the problem of random vibrations, and of random flights in one, two, three dimensions," *Philosophical Magazine*, Series 6, Vol. 37 (1919), pp. 321-347; R. Lüneburg, "Das Problem der Irrfahrt ohne Richtungsbeschränkung und die Randwertaufgabe der Potentialtheorie," *Mathematische Annalen*, Vol. 104 (1931), p. 700 etc.

Jessen (referred to on p. 316), where the densities are distributed along arbitrary convex curves. Applications to the ζ-function will be given later on.*

The Radial Distribution Function.

The modulus of (1) is, as (1), an almost-periodic function.† On the other hand, on replacing $z(t)$ by $|z(t)|$ we lose the linear independence of the frequences. It is nevertheless possible to calculate the distribution function of $|z(t)|$, i. e. the radial distribution function of $z(t)$. In fact, on placing

$$(87) \quad \nu(\xi) = 0, \ -\infty < \xi < 0; \quad \nu(\xi) = \int_0^{\xi} \eta\delta(\eta)\, d\eta/2\pi, \ 0 \leqq \xi < +\infty,$$

where δ is given by (83a) and (83b), it is easy to prove that ‡

$$(88) \qquad \mathfrak{M}(|z|^n) = \int_{-\infty}^{+\infty} \xi^n \, d\nu(\xi), \qquad (n = 0, 1, 2, \cdots).$$

Hence § $\nu(\xi)$ is the distribution function of $|z(t)|$. Thus the radial symmetry of $\phi(E)$ may be interpreted as an indication of the existence of a "mean motion" for the function $\arg z(t)$ although

$$\exp[i \arg z(t)] = z(t)/|z(t)|.$$

need not be almost-periodic.¶

We shall not use here all momentum equations (88) but only the relation

$$(89) \qquad \int_0^R r\delta(r)\, dr = 2\pi, \qquad (n = 0)$$

which is an obvious consequence of (67), (81), (79), (78) and (70).

* Cf. H. Bohr und R. Courant, "Neue Anwendungen der Theorie der diophantischen Approximationen auf die Riemannsche Zetafunktion," *Journal für Mathematik*, Vol. 144 (1914), pp. 249-274; H. Bohr und B. Jessen, "Ueber die Wertverteilung der Riemannschen Zetafunktion," *Acta Mathematica*, Vol. 54 (1930), pp. 1-35 and Vol. 58 (1932), pp. 1-55.

† This follows from the definition of the almost-periodicity inasmuch as

$$||z(t+a)| - |z(t)|| \leqq |z(t+a) - z(t)|$$

‡ The verification may be based upon the momentum identities developed in the Chapter on the Abelian integral equation.

§ Cf. *loc. cit. I* (referred to on p. 309).

¶ Cf. H. Weyl, "Sur une application de la théorie des nombres a la mécanique statistique et la théorie des perturbations," *L'Enseignement Mathématique*, Vol. 16 (1914), pp. 455-467. Cf. also F. Bernstein, "Ueber eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem," *Mathematische Annalen*, Vol. 71 (1912), pp. 417-439; and, on the other hand, H. Bohr, "Kleinere Beiträge zur Theorie der fastperiodischen Funktionen. I.," *Det Kgl. Danske Videnskabernes Selskab. Meddelelser*, Vol. 10, No. 10 (1930).

It is clear from (34), (82) and (89) that the second derivative of $\rho(\xi)$ is non-positive and not identically zero in a certain vicinity $R-\epsilon \leqq \xi \leqq R$ of the end-point $\xi = R$. It would be interesting to know if it is allowed to place $\epsilon = R$. This would mean that ρ represents, as does the Gauss curve, a so-called symmetrically convex distribution, i. e. one such that the density of probability is a non-increasing function of the distance from the origin. A detailed discussion of the curve ρ ought to be based * upon the Fourier integrals (84a), (84b).

A Property of Real Lagrangian Repartitions.

It has been pointed out in connection with (25) that the function $\rho(r)$ cannot be constant in the vicinity of points which are within the range $0 \leqq r \leqq R$. Also, the function $\rho(r)$ has derivatives of any order for all values of r. We now show that $\rho(r)$ need not be an analytic function in the range $0 \leqq r \leqq R$, even if $z(t)$ be analytic by virtue of its representation as an absolutely convergent Dirichlet series (cf. the Introduction).

Suppose that one of the partial vibrations of (1) or (5), say the first one $(j = 1)$, is "overwhelming" in the sense of Lagrange: †

$$r_1 > \sum_{j=2}^{\infty} r_j. \tag{90}$$

Then the density of probability $\rho'(r)$ belonging to $x(t)$ is a positive constant in the range

$$0 \leqq r \leqq r_1 - \sum_{j=2}^{\infty} r_j \, (< R), \tag{91}$$

without being a constant in the whole range $0 \leqq r \leqq R$, i. e. *the repartition of $x(t)$ is an equipartition in the domain* (91) *but not in the whole domain of $x(t)$*. This is, in reality, a consequence of (23) but the proof is shorter if we use $\delta(r)$.

First, from (1), (2) and (90),

$$|z(t)| \geqq r_1 - \sum_{j=2}^{\infty} r_j > 0,$$

i. e.

$$|z(t)| \geqq 2r_1 - R > 0; \qquad -\infty < t < +\infty.$$

* Cf. M. Mathias, "Ueber positive Fourier-Integrale," *Mathematische Zeitschrift*, Vol. 16 (1923), pp. 103-125.

† Cf. H. Bohr, "Das absolute Konvergenzproblem der Dirichletschen Reihen," *Acte Mathematica*, Vol. 36 (1913), pp. 202-209; A. Wintner, "Sur l'analyse anharmonique des inégalites séculaires fournies par l'approximation de Lagrange," *Rendiconti della R. Accademia Nazionale dei Lincei*, Series 6, Vol. 11 (1930), pp. 464-467.

Consequently, the distribution function $\phi(E)$ of $z(t)$ vanishes for all those rectangles E which are within the circle

$$x^2 + y^2 < (2r_1 - R)^2.$$

Hence from (81) and (79)

$$\delta(r) = 0 \quad \text{when} \quad 0 \leqq r \leqq 2r_1 - R,$$

or, according to (26),

$$\rho'(r) \equiv \rho'(0) \tag{92}$$

when $0 \leqq r \leqq 2r_1 - R$. On the other hand, (92) cannot hold in the *whole* range $0 \leqq r \leqq R$, i. e. the second derivative of $\rho(r)$ cannot be everywhere zero. This is obvious from (34) and (89). Finally, the constant (92) is, according to (26), equal to

$$2\int_0^R \delta(q)\,dq,$$

and, therefore, > 0 by (82) and (89).

ADDENDUM. (May 22, 1933). During the correction of the proof sheets, Jessen published in the *Zentralblatt für Mathematik und ihre Grenzgebiete*, Vol. 6 (May 10, 1933), pp. 162-163, a review of the author's paper *III*.

Jessen states that the remark in *III* regarding the example (4) is incorrect. In reality, my remark was "Diese Bedingung kann . . ." and not "Diese Bedingung muss . . ." so that Jessen's criticism is not justified.

Jessen states that although my method is a momentum method my results *loc. cit. III* are essentially the same as those of his Thesis, referred to above (p. 309). It is clear from the present paper that the analytical, viz. *explicit* methods, as developed *loc. cit. III*, yield essentially finer results than those of Jessen. Besides, Jessen does not treat the real-valued case, which was the exclusive topic of *III*, at all, and the connection between the real-valued and the complex-valued case is also not indicated by Jessen. Finally, the work of Bohr and Jessen on the zeta-function was *loc. cit III* not overlooked but exactly referred to.

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