BERRY-ESSEEN BOUNDS AND A THEOREM OF ERDÖS AND TURÁN ON UNIFORM DISTRIBUTION MOD 1

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1. Introduction. Let $\langle x_n \rangle$, $n = 1, 2, \cdots$, be a sequence of real numbers contained in $[0, 1)$. Denote by $A(N, x)$ the number of $n \leq N$ with $x_n < x$. The sequence $\langle x_n \rangle$ is called uniformly distributed (u.d.) if $N^{-1}A(N, x) \to x$ as $N \to \infty$ for all $0 \leq x \leq 1$. (In general, $\langle x_n \rangle$ is called u.d. mod 1 if the sequence of fractional parts $\{x_n\}$ is u.d.) It is easy to see that $\langle x_n \rangle$ is u.d. if and only if

$$D_N^* \overset{\text{def}}{=} \sup_{0 \leq x \leq 1} |N^{-1}A(N, x) - x| \to 0.$$

Another equivalent condition is the Weyl criterion: $\langle x_n \rangle$ is u.d. if and only if

$$S_N(h) \overset{\text{def}}{\equiv} N^{-1} \sum_{n \leq N} e^{2\pi i h x_n} \to 0$$

for all $h \in \mathbf{Z} - \{0\}$. (For the proof of the basic theorems see [5].) The following theorem due to Erdös and Turán [3] can be regarded as a quantitative version of the sufficient part of the Weyl criterion.

THEOREM A. *For any integer* $m \geq 1$

$$D_N^* \leq c_1 \frac{1}{m+1} + c_2 \sum_{h=1}^{m} \frac{1}{h} |S_N(h)|\cdot$$

The best constants c_1 and c_2 so far have been $c_1 = 17.2$ and $c_2 = 4.3$ (Niederreiter, unpublished). Much larger values were given by Yudin [9].

The purpose of this paper is to give various generalizations of this theorem and to point out their connections with other parts of analysis. We shall prove the following theorem.

THEOREM 1. *Let* $F(x)$ *be nondecreasing on* $[0, 1]$ *with* $F(0) = 0$ *and* $F(1) = 1$, *and let* $G(x)$ *satisfy a Lipschitz condition on* $[0, 1]$, *i.e.*,

$$|G(x) - G(y)| \leq M\,|x - y|$$

for all $0 \leq x, y \leq 1$. *Suppose that* $G(0) = 0$ *and* $G(1) = 1$. *Then for any positive integer* m

$$\sup_{0 \leq x \leq 1} |F(x) - G(x)| \leq \frac{4M}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1}\right) |\hat{F}(h) - \hat{G}(h)|$$

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where

$$\hat{F}(h) = \int_0^1 \exp(2\pi ihx)\, dF(x) \quad \textit{and} \quad \hat{G}(h) = \int_0^1 \exp(2\pi ihx)\, dG(x).$$

If we set $F(x) = N^{-1}A(N, x)$ and $G(x) = x$ for $0 \leq x \leq 1$, we get the Erdös-Turán theorem as a special case.

Theorem 1 has a celebrated continuous analogue, namely, the *Berry-Esseen inquality* [2; p. 206], [7; p. 285]. Let $F(x)$ be a distribution function of a random variable and let $G(x)$ satisfy a Lipschitz condition $|G(x) - G(y)| \leq M\,|x - y|$ for all $x, y \in \mathbf{R}$. Suppose that $G(-\infty) = 0$ and $G(\infty) = 1$. Then for any $T > 0$

$$\sup_{-\infty \leq x \leq \infty} |F(x) - G(x)| \leq \frac{24}{\pi}\frac{M}{T} + \frac{2}{\pi}\int_0^T \frac{|\hat{F}(t) - \hat{G}(t)|}{t}\, dt.$$

Remark. This is less restrictive than the usual assumption that $|G'(x)| \leq M$. The proofs given in [2] and [7] require no changes. Here and here only we put $\hat{F}(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$ and $\hat{G}(t) = \int_{-\infty}^{\infty} e^{itx}\, dG(x)$.

It is not difficult to modify the proof of Theorem 1 so as to give a theorem where the integral $\int_0^T$ is taken with respect to an arbitrary measure ν. But the only applications we could think of were the cases where ν is the Lebesgue measure yielding the Berry-Esseen inequality and where ν is the counting measure yielding Theorem 1. Consequently, we felt that the presentation of this generalization did not justify the extra effort.

Theorem 1 is also related to the sampling theorem in information theory [6].

Sampling theorem. Let $F(x)$ be of bounded variation on $[-\frac{1}{2}h, \frac{1}{2}h]$, $h > 0$. Suppose that the jumps of $F(x)$ at the endpoints of this interval are equal. Then

$$f(t) = \sum_{n=-\infty}^{\infty} f(n/h)\frac{\sin \pi(ht - n)}{\pi(ht - n)}, \qquad t \in \mathbf{R},$$

where

$$f(t) = \int_{[-\frac{1}{2}h, \frac{1}{2}h]} \exp(2\pi itx)\, dF(x).$$

Putting $h = 1$ we conclude that the "characteristic function" $f(t)$ of a random variable, bounded by $\frac{1}{2}$, is uniquely determined by its values $f(n)$ on the integers. (In this context the term characteristic function usually is reserved for the expression $\int_{-\infty}^{\infty} \exp(itx)\, dF(x)$. The factor 2π affects only a change in scale.) Shifting the scale half a unit to the left we see that this fact is also implied by Theorem 1. Of course, the condition on the boundedness of the random variable cannot be entirely omitted. Simply consider the sequence of independent Bernoulli trials centered at expectations, i.e., $P(X_k = -\frac{1}{2}) = P(X_k = \frac{1}{2}) = \frac{1}{2}$ for $k = 1, 2, \cdots$. The X_k have "characteristic function"

$$\hat{F}_k(t) = \int \exp(2\pi itX_k)\, dP = \cos \pi t.$$

But this coincides on the integers with $(\cos \pi t)^{2n+1}$, the "characteristic function" of $S_{2n+1} = X_1 + \cdots + X_{2n+1}$, which is a binomially distributed random variable, centered at expectation.

Since neither X_1 nor S_{2n+1} have a distribution function satisfying a Lipschitz condition, one could argue that this is the reason why Theorem 1 fails. But even if the distribution function $G(x)$ is assumed to satisfy a Lipschitz condition, Theorem 1 cannot be true unless the density $G'(x)$ has compact support. Simply consider the "characteristic function" $e^{-|t|}$ of a Cauchy-type random variable X and define a second one $\hat{F}(t)$ by interpolating linearly between the points with integer abscissae. By Polya's theorem [2; p. 169] $\hat{F}(t)$ is a "characteristic function" coinciding with $e^{-|t|}$ on the integers. But, obviously, $F(x)$ cannot be the distribution function of X.

We are fully aware that the following comments might make sense only after the reader has compared the proof of Theorem 1 with one of the standard proofs of the Berry-Esseen inequality (e.g., see Chung [2; pp. 206–208]). The following idea suggests itself immediately. Simply sum the formulae obtained from the integration by parts rather than integrate them as is done in some of the proofs of the Berry-Esseen inequality. That is, in essence, what we are doing. But the term with $h = 0$, if not properly disposed of, causes difficulties. These evaporate if we "center $F(x) - G(x)$ at expectation". A slightly more refined method will even yield a stronger result (see Theorem 1′).

Unfortunately, we were unable to make these methods work in dimension $s > 1$. By combining them with ideas from the original proof of the Erdös-Turán theorem the generalization to higher dimension can be carried out. We consider this to be the main result of the paper. In order to state the theorem we introduce some notation. Let $F(x_1, \cdots, x_s)$ be a distribution function on R^s with $F(1, \cdots, 1) = 1$ and $F(x_1, \cdots, x_s) = 0$ whenever some $x_j = 0, 1 \leq j \leq s$. Let $G(x_1, \cdots, x_s)$ be a function of bounded variation in the sense of Vitali on U^s, the closed s-dimensional unit cube, with $G(x_1, \cdots, x_s) = 0$ whenever some $x_j = 0, 1 \leq j \leq s$, and $G(1, \cdots, 1) = 1$. For any Borel set $B \subset U^s$ write $G(B) = \int_B dG$. We shall assume that for any rectangle $R = \{a_j \leq x_j \leq a_j + k_j, 1 \leq j \leq s\} \subset U^s$ we have

$$|G(a_j \leq x_j \leq a_j + k_j, 1 \leq j \leq s)| \leq M \prod_{j \leq s} k_j . \tag{1.1}$$

(These conditions are satisfied if G is a distribution function on U^s with density bounded by M.) We note once and for all that (1.1) implies a Lipschitz condition of the form

$$|G(x_1, \cdots, x_s) - G(y_1, \cdots, y_s)| \leq M \sum_{j \leq s} |x_j - y_j|$$

for $(x_1, \cdots, x_s), (y_1, \cdots, y_s) \in U^s$. For any integral vector $h = (h_1, \cdots, h_s)$ write $||h|| = \max_{1 \leq j \leq s} |h_j|$ and

$$R(h) = R(m, h) = \prod_{j \leq s} \left\{\left(1 - \frac{|h_j|}{m + 1}\right)^{-1} \max(|h_j|, 1)\right\}$$

where m is a given positive integer. Let $\langle h, x\rangle = \sum_{i\leq s} h_i x_i$ be the inner product of h and $x = (x_1, \cdots, x_s)$. Define $\hat{F}(h) = \hat{F}(h_1, \cdots, h_s) = \int_{U^s} \exp(2\pi i\langle h, x\rangle)\, dF(x)$ and similarly $\hat{G}(h)$.

THEOREM 2. *Under the above hypotheses, we have for any positive integer m*

$$\sup_{x\in U^s} |F(x) - G(x)| \ll_s \frac{M}{m+1} + \sum_{0<\|h\|\leq m} \frac{|\hat{F}(h) - \hat{G}(h)|}{R(h)}$$

where the constant implied by $\ll_s$ *depends on s only.*

Again we can specialize $F(x)$ and $G(x)$ to obtain the s-dimensional version of the Erdös-Turán theorem, due to Koksma [4] and Szüsz [8]. Unfortunately, our constants implied by $\ll_s$ are much worse than the best ones known for this particular case [5]. Similarly, there is a continuous analogue of Theorem 2, or an s-dimensional version of the Berry-Esseen inequality, due to von Bahr [1].

We are indebted to Professor Cambanis for providing us with reference [6].

2. The one-dimensional case. In this section we present the proof of Theorem 1 and some remarks concerning the values of the constants. In fact, we shall establish the following stronger version of Theorem 1.

THEOREM 1′. *Let* $F(x)$ *be nondecreasing on* $[0, 1]$ *with* $F(0) = 0$ *and* $F(1) = 1$, *and let* $G(x)$ *satisfy the Lipschitz condition*

$$|G(x) - G(y)| \leq M\, |x - y|$$

for all $0 \leq x, y \leq 1$. *Suppose also that* $G(0) = 0$ *and* $G(1) = 1$. *Then for any positive integer m*

$$\sup_{0\leq x,y\leq 1} |(F(x) - G(x)) - (F(y) - G(y))|$$

$$\leq \frac{4M}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)|\hat{F}(h) - \hat{G}(h)|.$$

If, in particular, $\langle x_n\rangle$ is a sequence in $[0, 1)$, then Theorem 1′ implies an inequality for the discrepancy D_N extended over all intervals mod 1. For $N \geq 1$ and an interval J mod 1 let $A(N, J)$ be the number of n, $1 \leq n \leq N$, with $x_n \in J$. Then define $D_N = \sup_J |N^{-1}A(N, J) - \lambda(J)|$ where $\lambda(J)$ is the length of J and the supremum is extended over all intervals J mod 1. Taking $F(x) = N^{-1}A(N, x)$ and $G(x) = x$ in Theorem 1′, we arrive at the following version of the Erdös-Turán theorem.

COROLLARY. *For any sequence* $\langle x_n\rangle$ *in* $[0, 1)$ *and any positive integer m we have*

$$D_N \leq \frac{4}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)|S_N(h)|.$$

In order to prove Theorem 1′ we need the following simple auxiliary result concerning the Fejér kernel.

LEMMA 1. *Let m be a positive integer, and let C and D be positive numbers with $D \leq \frac{1}{2}$ and $(m+1)D \geq C$. Then*

$$\int_D^{\frac{1}{2}} \frac{\sin^2 (m+1)\pi x}{\sin^2 \pi x}\, dx \leq \frac{1}{8D}\left(1 + \frac{1}{\pi C}\right). \tag{2.1}$$

Proof. Since $\sin \pi x \geq 2x$ for $0 \leq x \leq \frac{1}{2}$, we have

$$\int_D^{\frac{1}{2}} \frac{\sin^2 (m+1)\pi x}{\sin^2 \pi x}\, dx \leq \frac{1}{4}\int_D^{\frac{1}{2}} \frac{\sin^2 (m+1)\pi x}{x^2}\, dx. \tag{2.2}$$

Using integration by parts, we get

$$\begin{aligned}\int_D^{\frac{1}{2}} \frac{\sin^2 (m+1)\pi x}{x^2}\, dx &= \frac{1}{2D} + \frac{\sin 2(m+1)\pi D}{4(m+1)\pi D^2} - 1 - \int_D^{\frac{1}{2}} \frac{\sin 2(m+1)\pi x}{2(m+1)\pi x^3}\, dx \\ &\leq \frac{1}{2D} + \frac{1}{4(m+1)\pi D^2} + \int_D^{\frac{1}{2}} \frac{dx}{2(m+1)\pi x^3} \\ &\leq \frac{1}{2D} + \frac{1}{2(m+1)\pi D^2} \leq \frac{1}{2D}\left(1 + \frac{1}{\pi C}\right).\end{aligned}$$

In view of (2.2) we obtain (2.1).

Proof of Theorem 1′. It is convenient to extend $F(x)$ and $G(x)$ by setting $\bar{F}(x) = [x] + F(\{x\})$ and $\bar{G}(x) = [x] + G(\{x\})$ for $x \in \mathbf{R}$, where $[x]$ is the integral part and $\{x\}$ is the fractional part of x. We note that $\bar{F}(x)$ is nondecreasing on $\mathbf{R}$ and that $\bar{G}(x)$ satisfies a Lipschitz condition on $\mathbf{R}$ with constant M. We set $H(x) = \bar{F}(x) - \bar{G}(x)$ for $x \in \mathbf{R}$. Then $H(x)$ is periodic on $\mathbf{R}$ with period 1, and $\sup_{0 \leq x, y \leq 1} |H(x) - H(y)| = \sup_{x,y \in \mathbf{R}} |H(x) - H(y)|$.

We first consider the case where

$$\int_0^1 H(x)\, dx = 0. \tag{2.3}$$

For any integer h integration by parts yields

$$\hat{F}(h) - \hat{G}(h) = -2\pi i h \int_0^1 H(x) e^{2\pi i h x}\, dx. \tag{2.4}$$

Choose a positive integer m and a real number a to be determined later. Then (2.3) and (2.4) imply

$$\begin{aligned}&-\sum_{h=-m}^{m}{}^{*} (m + 1 - |h|) e^{-2\pi i h a} \frac{\hat{F}(h) - \hat{G}(h)}{2\pi i h} \\ &\qquad = \int_0^1 H(x)\left(\sum_{h=-m}^{m} (m + 1 - |h|) e^{2\pi i h (x-a)}\right) dx \\ &\qquad = \int_{-a}^{1-a} H(x+a)\left(\sum_{h=-m}^{m} (m + 1 - |h|) e^{2\pi i h x}\right) dx\end{aligned} \tag{2.5}$$

where the asterisk indicates that $h = 0$ is deleted from the range of summation.

Because of the periodicity of the integrand, the last integral may also be taken over $[-\frac{1}{2}, \frac{1}{2}]$. We note that

$$\sum_{h=-m}^{m} (m + 1 - |h|)e^{2\pi ihx} = \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x} \tag{2.6}$$

where the right-hand side is interpreted as $(m + 1)^2$ in case x is an integer. We infer from (2.5) and (2.6) that

$$\begin{aligned}\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} H(x + a) \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx\right| &\leq \frac{1}{2\pi} \sideset{}{^*}\sum_{h=-m}^{m} (m + 1 - |h|) \frac{|\hat{F}(h) - \hat{G}(h)|}{|h|} \\ &= \frac{1}{\pi} \sum_{h=1}^{m} (m + 1 - h) \frac{|\hat{F}(h) - \hat{G}(h)|}{h}.\end{aligned} \tag{2.7}$$

Let us set $\Delta = \sup_{x\in\mathbf{R}} |H(x)|$. Suppose that $\Delta > 2M/(m + 1)$, for otherwise the theorem is trivial. Since $H(x-) \leq H(x) \leq H(x+)$ for $x \in \mathbf{R}$, we have either $H(b-) = -\Delta$ or $H(b+) = \Delta$ for some $b \in \mathbf{R}$. We treat only the second alternative, the first one being almost identical. For $t > b$ we have $H(t) = \Delta + H(t) - H(b+) = \Delta + (\bar{F}(t) - \bar{F}(b+)) - (\bar{G}(t) - \bar{G}(b)) \geq \Delta - M(t - b)$. We set $D = \Delta/2M$ and choose $a = b + D$. Then from the above we get $H(x + a) \geq M(D - x)$ for $|x| < D$. In particular, we see that $D \leq \frac{1}{2}$, for otherwise $H(x)$ would be positive on an interval of length greater than 1, contradicting the fact that $H(n) = 0$ for $n \in \mathbf{Z}$. We obtain

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} &H(x + a) \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx \\ &= \left\{\int_{-D}^{D} + \int_{-\frac{1}{2}}^{-D} + \int_{D}^{\frac{1}{2}}\right\} H(x + a) \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx \\ &\geq M \int_{-D}^{D} (D - x) \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx \\ &\quad - 2MD \int_{-\frac{1}{2}}^{-D} \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx - 2MD \int_{D}^{\frac{1}{2}} \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx \\ &= 2MD \int_{0}^{D} \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx - 4MD \int_{D}^{\frac{1}{2}} \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx.\end{aligned} \tag{2.8}$$

The integral of the Fejér kernel over $[0, \frac{1}{2}]$ is $(m + 1)/2$ by (2.6). Therefore from (2.8) and Lemma 1

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} &H(x + a) \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx \\ &\geq 2MD \int_{0}^{\frac{1}{2}} \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx - 6MD \int_{D}^{\frac{1}{2}} \frac{\sin^2 (m + 1)\pi x}{\sin^2 \pi x}\, dx \\ &\geq \frac{m + 1}{2} \Delta - \tfrac{3}{4}M\left(1 + \frac{1}{\pi}\right) > \frac{m + 1}{2} \Delta - M\end{aligned}$$

where we used $(m + 1)D > 1$. Combining the above inequality with (2.7), we arrive at

$$\Delta \leq \frac{2M}{m+1} + \frac{2}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1}\right) |\hat{F}(h) - \hat{G}(h)|. \tag{2.9}$$

In particular, this proves the theorem in case (2.3) is satisfied.

We consider now the case where $\int_0^1 H(x)\,dx \neq 0$. We shall first show that there exists c, $0 < c < 1$, such that

$$\int_0^1 H(x)\,dx = H(c). \tag{2.10}$$

Suppose first $\int_0^1 H(x)\,dx > 0$. There exists s, $0 < s < 1$, such that $H(s) \geq \int_0^1 H(x)\,dx$. We note also that $H(1) = 0$. Since the only discontinuities of $H(x)$ are positive jumps, the function $H(x)$ must attain the value $\int_0^1 H(x)\,dx$ in the interval $[s, 1)$. If $\int_0^1 H(x)\,dx < 0$, one proceeds analogously.

With a number c satisfying (2.10), we define $F_1(x) = \bar{F}(x + c) - \bar{F}(c)$ and $G_1(x) = \bar{G}(x + c) - \bar{G}(c)$ for $x \in \mathbf{R}$. We observe that $F_1(x)$ and $G_1(x)$ share the properties of $\bar{F}(x)$ and $\bar{G}(x)$ respectively. Furthermore, we have $\int_0^1 (F_1(x) - G_1(x))\,dx = \int_0^1 (H(x + c) - H(c))\,dx = 0$. By what we have already shown, the inequality

$$\sup_{x\in\mathbf{R}} |F_1(x) - G_1(x)| \leq \frac{2M}{m+1} + \frac{2}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1}\right) \left|\int_0^1 e^{2\pi i h x}\, d(F_1(x) - G_1(x))\right| \tag{2.11}$$

holds for any positive integer m. We note that

$$\begin{aligned}\left|\int_0^1 e^{2\pi i h x}\, d(F_1(x) - G_1(x))\right| &= \left|\int_0^1 e^{2\pi i h x}\, d(\bar{F}(x + c) - \bar{G}(x + c))\right| \\ &= \left|\int_c^{1+c} e^{2\pi i h (x-c)}\, dH(x)\right| = \left|\int_c^{1+c} e^{2\pi i h x}\, dH(x)\right| \\ &= \left|\int_0^1 e^{2\pi i h x}\, dH(x)\right| = |\hat{F}(h) - \hat{G}(h)|\end{aligned}$$

for all integers h. It follows from (2.11) that

$$\sup_{x,y\in\mathbf{R}} |(F_1(x) - G_1(x)) - (F_1(y) - G_1(y))| \leq \frac{4M}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1}\right) |\hat{F}(h) - \hat{G}(h)|$$

and so

$$\sup_{x,y\in\mathbf{R}} |H(x + c) - H(y + c)| \leq \frac{4M}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1}\right) |\hat{F}(h) - \hat{G}(h)|$$

which is the desired inequality.

An alternative (and somewhat shorter) proof of the weaker Theorem 1 rests on centering $H(x)$ at expectation, i.e., replacing the function $H(x)$ in the above argument by $H^*(x) = H(x) - I$, where $I = \int_0^1 H(x)\,dx$. In exactly the same way, one proves then

$$\sup_{x\in\mathbf{R}} |H(x) - I| \leq \frac{2M}{m+1} + \frac{2}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)|\hat{F}(h) - \hat{G}(h)|.$$

Choosing $x = 0$, one obtains an estimate for $|I|$ which implies the desired estimate for $\sup_{x\in\mathbf{R}} |H(x)|$.

We add some remarks concerning the values of the constants in the corollary of Theorem 1′. Suppose c_1 and c_2 are absolute constants such that $D_N \leq c_1/(m+1) + c_2 \sum_{h=1}^{m} (1/h - 1/(m+1))\,|S_N(h)|$ holds for any sequence $\langle x_n \rangle$ in $[0, 1)$ and any positive integer m. Then choosing $N = 2$, $x_1 = 0$, and $x_2 = \frac{1}{2}$, we get

$$\frac{1}{2} \leq \frac{c_1}{m+1} + c_2 \sum_{\substack{h\leq m\\ h \text{ even}}} \left(\frac{1}{h} - \frac{1}{m+1}\right).$$

With $m = 1$ we obtain $c_1 \geq 1$, and with $m = 3$ we obtain $c_1 + c_2 \geq 2$. Also, taking $m = 19$ and using $c_1 \leq 4$, we get $c_2 \geq 0.31$.

3. Proof of Theorem 2 for $s = 2$. We shall prove Theorem 2 in detail for $s = 2$ only and indicate the necessary changes for $s \geq 3$ in the next section.

Write $H(x, y) = F(x, y) - G(x, y)$, $\hat{H}(h_1, h_2) = \hat{F}(h_1, h_2) - \hat{G}(h_1, h_2)$ and define D by $\Delta = \sup_{(x,y)\in U^2} |H(x, y)| = 100\,MD$.

LEMMA 2. *If* $(u, v) \in U^2$ *satisfies* $H(u, v) = \Delta$ *and* $\max(u, v) \geq 1 - 50D$, *then the conclusion of Theorem* 2 *holds.*

Proof. Without loss of generality we assume that $\max(u, v) = v$. Then

$$\begin{aligned} H(u, 1) &= \Delta + F(u, 1) - F(u, v) - (G(u, 1) - G(u, v)) \\ &\geq \Delta - 50MD = \tfrac{1}{2}\Delta. \end{aligned}$$

But $F(x, 1)$ and $G(x, 1)$ satisfy the hypotheses of Theorem 1. Hence

$$\begin{aligned} \tfrac{1}{2}\Delta \leq H(u, 1) &\leq \sup_{0\leq x\leq 1} |H(x, 1)| \\ &\leq \frac{4M}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(1 - \frac{h}{m+1}\right)\frac{1}{h}\left|\int_0^1 \exp(2\pi i h x)\,dH(x, 1)\right|. \end{aligned} \tag{3.1}$$

Since the last integral is just $\hat{H}(h, 0)$, the lemma is proved.

LEMMA 3. *Let* $0 < \epsilon \leq 1$. *Then* $H(u-, v-) \leq -\epsilon\Delta$ *where* $(u, v) \in U^2$ *implies* $\min(u, v) \geq 100\epsilon D$.

Proof. In fact, $u < 100\epsilon D$ leads to a contradiction, $-100\epsilon MD = -\epsilon\Delta \geq F(u-, v-) - (G(u, v) - G(0, v)) > -100\epsilon MD$.

COROLLARY. *We may assume that* $D \leq 1/50$.

Proof. If $D > 1/50$, an application of Lemma 3 with $\epsilon = 1$ shows that there is no point $(u, v) \in U^2$ with $H(u-, v-) = -\Delta$. Hence there must exist a point $(u, v) \in U^2$ with $H(u, v) = \Delta$. But then, since $1 - 50D < 0 \leq \max(u, v)$, Lemma 2 applies.

Define

$$f(t) = \begin{cases} \dfrac{\sin^2 \pi(m+1)t}{\sin^2 \pi t} & \text{for} \quad t \notin \mathbf{Z} \\ (m+1)^2 & \text{for} \quad t \in \mathbf{Z}. \end{cases}$$

Then by (2.6)

$$f(t_1)f(t_2) = \sum_{\|h\| \leq m} (m + 1 - |h_1|)(m + 1 - |h_2|) \exp(2\pi i(h_1 t_1 + h_2 t_2)).$$

For $0 \leq \alpha \leq 1$ put

$$r_\alpha(w) = \frac{1}{m+1} \int_0^\alpha f(w - t)\, dt.$$

We observe that for $0 \leq \alpha, \beta, a, b \leq 1$ we have

$$r_\alpha(x - a)r_\beta(y - b) = \sum_{\|h\| \leq m} a_{h_1}(\alpha, a)a_{h_2}(\beta, b) \exp(2\pi i(h_1 x + h_2 y))$$

with $a_0(\alpha, a) = \alpha$ and

$$a_{h_1}(\alpha, a) = \frac{1}{2\pi i h_1} \frac{m + 1 - |h_1|}{m + 1} (1 - e^{-2\pi i h_1 \alpha})e^{-2\pi i h_1 a} \text{ for } h_1 \neq 0.$$

Using the fact $\int_{U^2} dH(x, y) = 0$, we obtain for any choice of α, β, a and b

$$\begin{aligned} \left| \int_{U^2} r_\alpha(x - a)r_\beta(y - b)\, dH(x, y) \right| &= \Big| \sum_{0 < \|h\| \leq m} a_{h_1}(\alpha, a)a_{h_2}(\beta, b)\hat{H}(h_1, h_2) \Big| \\ &\leq \frac{1}{\pi} \sum_{0 < \|h\| \leq m} \frac{|\hat{H}(h)|}{R(h)}. \end{aligned} \tag{3.2}$$

Our goal is to estimate the integral from below by a nontrivial linear combination of Δ, $M/(m + 1)$ and the last sum in (3.2). Integration by parts gives

$$\begin{aligned} \int_{U^2} r_\alpha(x - a)r_\beta(y - b)\, dH(x, y) = &-r_\beta(1 - b) \int_0^1 H(x, 1)r'_\alpha(x - a)\, dx \\ &- r_\alpha(1 - a) \int_0^1 H(1, y)r'_\beta(y - b)\, dy \\ &+ \int_{U^2} H(x, y)r'_\alpha(x - a)r'_\beta(y - b)\, dx\, dy. \end{aligned} \tag{3.3}$$

Notice that $0 \leq r_\alpha(1 - a), r_\beta(1 - b) \leq 1$ for any choice of α, β, a and b.

LEMMA 4. *For any choice of α, β, a and b we have*

$$\left|\int_0^1 H(x, 1)r'_\alpha(x - a)\, dx\right| \leq \frac{8M}{m+1} + \frac{8}{\pi}\sum_{h=1}^{m}\frac{|\hat{H}(h, 0)|}{R(h, 0)}$$

and

$$\left|\int_0^1 H(1, y)r'_\beta(y - b)\, dy\right| \leq \frac{8M}{m+1} + \frac{8}{\pi}\sum_{h=1}^{m}\frac{|\hat{H}(0, h)|}{R(0, h)}.$$

Proof. We prove the first inequality only. Its left-hand side equals

$$\frac{1}{m+1}\left|\int_0^1 H(x, 1)(f(x - a) - f(x - a - \alpha))\, dx\right|$$

$$\leq 2 \sup_{0 \leq x \leq 1} |H(x, 1)| \frac{1}{m+1}\int_0^1 f(x)\, dx = 2 \sup_{0 \leq x \leq 1} |H(x, 1)|$$

$$\leq \frac{8M}{m+1} + \frac{8}{\pi}\sum_{h=1}^{m}\left(1 - \frac{h}{m+1}\right)\frac{|\hat{H}(h, 0)|}{h}$$

by (3.1) and the remark following it.

LEMMA 5. *If $(u, v) \in U^2$ satisfies $H(u, v) \geq \frac{1}{4}\Delta$ and* $\max(u, v) \leq 2D$, *then the conclusion of Theorem* 2 *holds.*

Proof. Choose $a = b = 0$. For $0 \leq x \leq \alpha - 1/2(m + 1)$ we have

$$r_\alpha(x) = \frac{1}{m+1}\int_0^\alpha f(x - t)\, dt \geq \frac{1}{m+1}\int_0^{1/(2m+2)} f(t)\, dt \geq 2/\pi^2.$$

Hence

$$\int_{U^2} r_\alpha(x)r_\beta(y)\, dF(x, y) \geq \frac{4}{\pi^4} F\left(\alpha - \frac{1}{2(m+1)}, \beta - \frac{1}{2(m+1)}\right).$$

Next choose $\alpha = u + 1/2(m + 1)$ and $\beta = v + 1/2(m + 1)$. Since we may assume $1/(m + 1) \leq D$, we have $0 \leq \alpha, \beta \leq 1/20$ in view of the corollary of Lemma 3. Thus

$$F(u, v) \leq \tfrac{1}{4}\pi^4 \int_{U^2} r_\alpha(x)r_\beta(y)\, dF(x, y)$$

and

$$\begin{aligned}\tfrac{1}{4}\Delta \leq H(u, v) &= F(u, v) - G(u, v)\\ &\leq \frac{\pi^4}{4}\int_{U^2} r_\alpha(x)r_\beta(y)\, dH(x, y) + \frac{\pi^4}{4}\int_{U^2} r_\alpha(x)r_\beta(y)\, dG(x, y) - G(u, v).\end{aligned}$$

The first integral has been estimated in (3.2). Because of (1.1) the second integral is bounded by

$$M \int_{U^2} r_\alpha(x) r_\beta(y)\, dx\, dy = M\alpha\beta \leq \frac{1}{20} M\left(2D + \frac{1}{2(m+1)}\right).$$

Finally, $|G(u, v)| = |G(u, v) - G(u, 0)| \leq 2MD$. Hence

$$\frac{1}{4}\Delta \leq \frac{1}{4}\pi^3 \sum_{0<||h||\leq m} \frac{|\hat{H}(h)|}{R(h)} + \frac{1}{4}\pi^4\left(10^{-3}\Delta + \frac{1}{40}\frac{M}{m+1}\right) + \frac{1}{50}\Delta$$

or

$$\Delta \leq \frac{5}{4}\pi^3 \sum_{0<||h||\leq m} \frac{\hat{H}(h)}{R(h)} + 4\frac{M}{m+1}.$$

In view of (3.3), Lemma 4 and the observation made after (3.2) we have to estimate the last integral in (3.3) from below. In these estimates we frequently have to deal with integrals of the following type.

LEMMA 6. *We have*

$$A = \int_{\max(|x|,|y|)\leq D} f(x)f(y)\, dx\, dy > (m+1)^2 - \frac{m+1}{D}$$

$$B = \int_{D<\max(|x|,|y|)\leq \frac{1}{2}} f(x)f(y)\, dx\, dy < \frac{m+1}{D}$$

and

$$A + B = (m+1)^2.$$

Proof. The second integral is bounded by

$$4\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\, dx \int_D^{\frac{1}{2}} f(y)\, dy \leq 4(m+1)\int_D^{\frac{1}{2}} \frac{dx}{4x^2} < \frac{m+1}{D}.$$

The estimate for A follows now from the last identity.

LEMMA 7. *Suppose $H(u-, v-) \leq -\frac{1}{2}\Delta$ for some $(u, v) \in U^2$. Then the conclusion of Theorem 2 holds.*

Proof. Observe that for $x \leq u$ and $y \leq v$

$$\begin{aligned} H(x, y) &\leq -50MD + (F(x, y) - F(u-, v-)) - (G(x, y) - G(u, v)) \\ &\leq -50MD + M(u - x + v - y). \end{aligned}$$

Choose $a = u - D$ and $b = v - D$ and observe that $(a, b) \in U^2$ since $\min(u, v) \geq 50D$ by Lemma 3. Hence

$$H(x, y) \leq -M(48D + (x - a) + (y - b)) \tag{3.4}$$

for $(x, y) \in \square = \square(a, b) \subset U^2$. Here we introduced the notation

$$\square(\xi, \eta) = \{(x, y) : \max(|x - \xi|, |y - \eta|) \leq D\} \text{ and } \square^c = U^2 \backslash \square. \tag{3.5}$$

Now

$$
\begin{aligned}
(m+1)^2 \int_{U^2} H(x, y) r'_\alpha(x-a) r'_\beta(y-b)\, dx\, dy
&= \int_{U^2} H(x, y) f(x-a) f(y-b)\, dx\, dy \\
&\quad + \int_{U^2} H(x, y) f(x-a-\alpha) f(y-b-\beta)\, dx\, dy \\
&\quad - \int_{U^2} H(x, y) f(x-a) f(x-b-\beta)\, dx\, dy \\
&\quad - \int_{U^2} H(x, y) f(x-a-\alpha) f(y-b)\, dx\, dy \\
&= I + II - III - IV.
\end{aligned} \tag{3.6}
$$

Using (3.4) we obtain

$$
\begin{aligned}
I &\le -M \int_{\square} (48D + (x-a) + (y-b)) f(x-a) f(y-b)\, dx\, dy \\
&\quad + \Delta \int_{\square^c} f(x-a) f(y-b)\, dx\, dy \\
&= -48MDA + \Delta B.
\end{aligned} \tag{3.7}
$$

We choose $\alpha = 1 - u + 2D$ and $\beta = 1 - v + 2D$ and observe that $0 \le \alpha, \beta \le 1$ since $\min(u, v) \ge 50D$, as noted before.

For the estimate of II we may assume that on $\square = \square(D, D)$ we have $H(x, y) < \frac{1}{4}\Delta$, for otherwise we are done by Lemma 5. Hence by the periodicity of f and Lemma 6

$$
\begin{aligned}
II &\le \int_{\square} 25MDf(x-1-D) f(y-1-D)\, dx\, dy \\
&\quad + \Delta \int_{\square^c} f(x-1-D) f(y-1-D)\, dx\, dy \\
&\le 25MDA + \Delta B.
\end{aligned} \tag{3.8}
$$

Next, we observe that on $\square = \square(a, D)$, we have $H(x, y) = F(x, y) - G(x, y) + G(x, 0) \ge -2MD$ and hence

$$
\begin{aligned}
III &\ge -2MD \int_{\square} f(x-a) f(y-1-D)\, dx\, dy \\
&\quad - \Delta \int_{\square^c} f(x-a) f(y-1-D)\, dx\, dy \\
&= -2MDA - \Delta B.
\end{aligned} \tag{3.9}
$$

By symmetry we get the same bound for IV. Putting (3.6)–(3.9) together and using Lemma 6 we obtain $\int_{U^2} H(x, y) r'_\alpha(x-a) r'_\beta(y-b)\, dx\, dy \le -19MD + 419M/(m+1) \le 0$ since otherwise the lemma is trivially true. Hence we conclude from (3.2), (3.3) and Lemma 4 that $19MD \le 435M/(m+1) + (9/\pi)\cdot \sum_{0<\|h\|\le m} |\hat{H}(h)|/R(h)$ which gives the result.

Finally, we can finish the proof of the theorem. We can assume that there exists a point $(u, v) \in U^2$ with $H(u, v) = \Delta$, because if $H(u_1-, v_1-) = -\Delta$ for some $(u_1, v_1) \in U^2$, then Lemma 7 applies. Actually, and for the same reason, we even assume that $H(x, y) > -\frac{1}{2}\Delta$ for all $(x, y) \in U^2$ and that $\max(u, v) < 1 - 50D$, since otherwise we were done by Lemma 2. The remainder of the proof is nearly the same as the proof in Lemma 7. We have $H(x, y) = \Delta + F(x, y) - F(u, v) - G(x, y) + G(u, v) \ge \Delta - M(x - u + y - v)$ for all $x \ge u$ and $y \ge v$. Choose $a = u + D$ and $b = v + D$. Then

$$H(x, y) \ge M(98D - (x - a) - (y - b)) \tag{3.10}$$

for $(x, y) \in \square = \square(a, b) \subset U^2$. Using the same notation as in (3.6) we obtain

$$\begin{aligned} I &\ge M \int_{\square} (98D - (x - a) - (y - b)) f(x - a) f(y - b)\, dx\, dy \\ &\quad - \tfrac{1}{2}\Delta \int_{\square^c} f(x - a) f(y - b)\, dx\, dy \\ &= 98\, MDA - \tfrac{1}{2}\Delta B\cdot \end{aligned}$$

By the assumption on $H(x, y)$

$$II \ge -\tfrac{1}{2}\Delta(m + 1)^2.$$

Next, choose $\alpha = 1 - u - 2D > 0$ and $\beta = 1 - v - 2D > 0$ as $\max(u, v) < 1 - 50D$. Let $\square = \square(a, 1 - D)$ and $\sigma = u + 2D < 1$. Then $H(x, y) = H(\sigma, 1) + F(x, y) - F(\sigma, 1) - G(x, y) + G(\sigma, 1) \le |H(\sigma, 1)| + 4MD$ for $(x, y) \in \square$. Thus

$$\begin{aligned} III &\le \int_{\square} (|H(\sigma, 1)| + 4MD) f(x - a) f(y - 1 + D)\, dx\, dy \\ &\quad + \Delta \int_{\square^c} f(x - a) f(y - 1 + D)\, dx\, dy \\ &= (|H(\sigma, 1)| + 4MD)A + \Delta B. \end{aligned}$$

Similarly with $\tau = v + 2D$ we get

$$IV \le (|H(1, \tau)| + 4MD)A + \Delta B.$$

We substitute the estimates for I–IV into (3.6) and obtain, using Lemma 6,

$$\begin{aligned} &\int_{U^2} H(x, y) r'(x - a) r'(y - b)\, dx\, dy \\ &\qquad \ge 40MD - 340M/(m + 1) - |H(\sigma, 1)| - |H(1, \tau)| \\ &\qquad \ge 40MD - 348M/(m + 1) - (4/\pi) \sum_{0<\|h\|\le m} \frac{|\hat{H}(h)|}{R(h)} \end{aligned} \tag{3.11}$$

by (3.1) and the remark following it. The last term can be assumed to be positive since otherwise the theorem is trivially true. Hence by (3.2), (3.3) and Lemma 4 we obtain $40MD \leq 364M/(m+1) + (13/\pi) \sum_{0<\|h\|\leq m} |\hat{H}(h)|/R(h)$ which implies the result.

4. The general case. In the present section we shall sketch the proof of Theorem 2 for $s \geq 3$. We assume throughout that Theorem 2 holds for dimension $k < s$ and use induction.

Define D by $\Delta = \sup_{x\in U^s} |H(x)| = 10^s MD$. We replace Lemma 2 by the following lemma.

LEMMA 2s. *Let* $10^{-s} \leq \epsilon \leq 1$. *If* $(u_1, \cdots, u_s) \in U^s$ *satisfies* $H(u_1, \cdots, u_s) \geq \epsilon\Delta$ *and* $\max u_j \geq 1 - \frac{1}{2}10^s \epsilon D$, *then the conclusion of Theorem 2 holds.*

LEMMA 3s. *Let* $0 < \epsilon \leq 1$. *Then* $H(u_1-, \cdots, u_s-) \leq -\epsilon\Delta$, *where* $(u_1, \cdots, u_s) \in U^s$, *implies* $\min u_j \geq 10^s \epsilon D$.

COROLLARY s. *We may assume that* $D \leq 2\cdot 10^{-s}$.

We use again the functions r_α considered in Section 3. Then for $0 \leq \alpha_j$, $a_j \leq 1$, $1 \leq j \leq s$, we have

(4.1)

$$\left|\int_{U^s} \prod_{j\leq s} r_{\alpha_j}(x_j - a_j)\, dH(x_1, \cdots, x_s)\right| = \Big| \sum_{0<\|h\|\leq m} \prod_{j\leq s} a_{h_j}(\alpha_j)\cdot \hat{H}(h_1, \cdots, h_s)\Big| \leq \frac{1}{\pi} \sum_{0<\|h\|\leq m} \frac{|\hat{H}(h)|}{R(h)}.$$

Integrating by parts we obtain

$$(4.2) \quad \left|\int_{U^s} \prod_{j\leq s} r_{\alpha_j}(x_j - a_j)\, dH(x_1, \cdots, x_s)\right| \geq \left|\int_{U^s} H(x_1, \cdots, x_s) \prod_{j\leq s} r'_{\alpha_j}(x_j - a_j)\, dx_1 \cdots dx_s\right| - \sum_{\nu=1}^{s-1} \sum_{1\leq j_1<\cdots<j_\nu\leq s} |I^{(\nu)}_{j_1\cdots j_\nu}|$$

with

$$I^{(\nu)}_{j_1\cdots j_\nu} = \int_{U^\nu} H(\cdots) \prod_{k\leq \nu} r'_{\alpha_{j_k}}(x_{j_k} - a_{j_k})\, dx_{j_1} \cdots dx_{j_\nu}, \quad 1 \leq j_1 < \cdots < j_\nu \leq s.$$

Here in $H(\cdots)$ the coordinates x_n with $n \neq j_k$, $1 \leq k \leq \nu$, are replaced by 1. Lemma 4 becomes the next lemma.

LEMMA 4s. *For any choice of* α_j, a_j, $1 \leq j \leq s$, *we have*

$$|I^{(\nu)}_{j_1\cdots j_\nu}| \leq 2^\nu\left(A_\nu M/(m+1) + B_\nu \sum_{0<\|h\|\leq m} \frac{|\hat{H}(h)|}{R(h)}\right)$$

for $1 \leq j_1 < \cdots < j_\nu \leq s$.

LEMMA 5s. *Suppose that there is a point* $(u_1, \cdots, u_s) \in U^s$ *with* $H(u_1, \cdots, u_s) \geq 10^{-s+1}\Delta$ *and having at least two of the coordinates* u_j *bounded by* $2D$. *Then the conclusion of Theorem* 2 *holds.*

Proof. We can assume that $D \geq 1/(m+1)$ since otherwise there is nothing to prove. We choose $a_j = 0$, $\alpha_j = u_j + 1/2(m+1)$, $1 \leq j \leq s$, and observe that $\alpha_j \leq 1$, $1 \leq j \leq s$. In fact, if $u_j > 1 - 1/2(m+1) \geq 1 - \frac{1}{2}D$, Lemma 2s applies with $\epsilon = 10^{-s}$. Hence, as in Lemma 5

$$\begin{aligned} 10^{-s+1}\Delta \leq \left(\frac{\pi^2}{2}\right)^s \int_{U^s} \prod_{j \leq s} r_{\alpha_j}(x_j - a_j)\, dH(x_1, \cdots, x_s) \\ + \left(\frac{\pi^2}{2}\right)^s \int_{U^s} \prod_{j \leq s} r_{\alpha_j}(x_j - a_j)\, dG(x_1, \cdots, x_s) - G(u_1, \cdots, u_s). \end{aligned} \tag{4.3}$$

The first integral is estimated in (4.1). In view of (1.1) the second integral does not exceed

$$\begin{aligned} M \cdot \prod_{j \leq s} \alpha_j &\leq M\left(2D + \frac{1}{2(m+1)}\right)^2 \\ &\leq 5M \cdot 10^{-s}\left(2D + \frac{1}{2(m+1)}\right) \end{aligned}$$

by Corollary s and the assumption on D. Finally, by (1.1), $|G(u_1, \cdots, u_s)| \leq 4MD^2 \leq 10^{-s+1}MD$. Substituting these inequalities into (4.3) we obtain the result.

LEMMA 6s. *We have*

$$A = \int_{\max|x_j| \leq D} f(x_1) \cdots f(x_s)\, dx_1 \cdots dx_s > (m+1)^s - (m+1)^{s-1}\frac{s}{2D}$$

$$B = \int_{D < \max|x_j| \leq \frac{1}{2}} f(x_1) \cdots f(x_s)\, dx_1 \cdots dx_s < (m+1)^{s-1}\frac{s}{2D}$$

and

$$A + B = (m+1)^s.$$

LEMMA 7s. *Suppose that* $H(u_1-, \cdots, u_s-) \leq -4^{-s+1}\Delta$ *for some* $(u_1, \cdots, u_s) \in U^s$; *then the conclusion of the theorem holds.*

Proof. Write

$$\square = \square(\xi_1, \cdots, \xi_s) = \{(x_1, \cdots, x_s) : \max|x_j - \xi_j| \leq D\}, \quad \square^c = U^s \backslash \square,$$

and choose $a_j = u_j - D$ and $\alpha_j = 1 - u_j + 2D$, $1 \leq j \leq s$. It follows from Lemma 3s that $0 \leq a_j, \alpha_j \leq 1$ for $1 \leq j \leq s$. Now

$$H(x_1, \cdots, x_s) \leq -4^{-s+1}\Delta + sMD + M\sum_{j \leq s}(a_j - x_j) \tag{4.4}$$

for $(x_1, \cdots, x_s) \in \square\ (a_1, \cdots, a_s)$. Expanding

$$(4.5) \qquad (m+1)^s \int_{U^s} H(x_1, \cdots, x_s) r'_{\alpha_1}(x_1 - a_1) \cdots r'_{\alpha_s}(x_s - a_s)\, dx_1 \cdots dx_s$$

in the same way as in (3.6), we observe that there are exactly 2^{s-1} terms of positive sign and 2^{s-1} terms of negative sign. Among those with positive sign we single out the term $\int_{U^s} H(x_1, \cdots, x_s) f(x_1 - a_1) \cdots f(x_s - a_s)\, dx_1 \cdots dx_s$. By (4.4) it does not exceed

$$(4.6) \qquad (-4^{-s+1}\Delta + sMD)A + \Delta B.$$

Each of the other $2^{s-1} - 1$ terms with positive sign contains at least two factors of the form $f(x_j - 1 - D) = f(x_j - D)$. For the estimation of these terms we choose $\square = \square\ (\xi_1, \cdots, \xi_s)$ with $\xi_j = D$ for every j corresponding to a factor $f(x_j - D)$ and $\xi_j = a_j$ otherwise. By Lemma 5s we can assume that $H(x_1, \cdots, x_s) < 10^{-s+1}\Delta$ for $(x_1, \cdots, x_s) \in \square\ (\xi_1, \cdots, \xi_s)$. This leads to the upper bound for the $2^{s-1} - 1$ "positive terms"

$$(4.7) \qquad 10^{-s+1}\Delta A + \Delta B.$$

Each term with negative sign contains at least one factor $f(x_j - 1 - D) = f(x_j - D)$. We choose $\square\ (\xi_1, \cdots, \xi_s)$ as before and observe that $H(x_1, \cdots, x_s) \geq -2MD$ for $(x_1, \cdots, x_s) \in \square\ (\xi_1, \cdots, \xi_s)$. This leads to the lower bound for the 2^{s-1} "negative terms"

$$(4.8) \qquad -2MDA - \Delta B.$$

Putting (4.5)–(4.8) together we obtain

$$\int_{U^s} H(x_1, \cdots, x_s) r'_{\alpha_1}(x_1 - a_1) \cdots r'_{\alpha_s}(x_s - a_s)\, dx_1 \cdots dx_s \leq -\tfrac{1}{2}4^{-s}\Delta + s20^s M/(m+1) \leq 0.$$

The result follows now in the usual way from (4.1), (4.2) and Lemma 4s.

To finish the proof we can assume that there exists a point $(u_1, \cdots, u_s) \in U^s$ with $H(u_1, \cdots, u_s) = \Delta$, that $H(x_1, \cdots, x_s) > -4^{-s+1}\Delta$ and that $\max u_j \leq 1 - \frac{1}{2}10^s D$.

Choose $a_j = u_j + D$, $\alpha_j = 1 - u_j - 2D$ and treat

$$(m+1)^s \int_{U^s} H(x_1, \cdots, x_s) r'_{\alpha_1}(x_1 - a_1) \cdots r'_{\alpha_s}(x_s - a_s)\, dx_1 \cdots dx_s$$

as before. We obtain

$$(4.9) \qquad \int_{U^s} H(x_1, \cdots, x_s) f(x_1 - a_1) \cdots f(x_s - a_s)\, dx_1 \cdots dx_s \geq (\Delta - sMD)A - \Delta B.$$

For each of the $2^{s-1} - 1$ remaining "positive terms" we get the lower bound

$$-4^{-s+1}\Delta(m + 1)^s. \tag{4.10}$$

For the estimate of the "negative terms" we again observe that each of them contains at least one factor $f(x_j - 1 + D) = f(x_j + D)$ and accordingly we choose $\square(\xi_1, \cdots, \xi_s)$ with $\xi_j = 1 - D$ for each j corresponding to such a factor and $\xi_j = a_j$ otherwise. Moreover, put $\sigma_j = 1$ if $\xi_j = 1 - D$ and $\sigma_j = u_j + 2D$ otherwise. Then a typical "negative term" is bounded by

$$(|H(\sigma_1, \cdots, \sigma_s)| + 2sMD)A + \Delta B. \tag{4.11}$$

The result follows now from (4.9)–(4.11), (4.1), (4.2), Lemma 4s and the induction hypothesis.

Added in proof. In the one-dimensional case, similar inequalities have been shown by Faĭnleĭb, Izv. Akad. Nauk SSSR Ser. Mat., vol. 32(1968), pp. 859–879, and Elliott, J. Number Th., vol. 4(1972), pp. 509–522.

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