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On Riemann "Nondifferentiable" Function and Schrödinger Equation

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Abstract—The function $\psi := \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{\pi i(tn^2+2xn)}/(\pi i n^2)$, $\{t,x\}\in\mathbb{R}^2$, is studied as a (generalized) solution of the Cauchy initial value problem for the Schrödinger equation. The real part of the restriction of ψ on the line $x=0$, that is, the function $R := \operatorname{Re}\psi|_{x=0} = \frac{2}{\pi}\sum_{n\in\mathbb{N}}\frac{\sin\pi n^2 t}{n^2}$, $t\in\mathbb{R}$, was suggested by B. Riemann as a plausible example of a continuous but nowhere differentiable function. The points are established on $\mathbb{R}^2$ where the partial derivative $\frac{\partial\psi}{\partial t}$ exists and equals -1. These points constitute a countable set of open intervals parallel to the x-axis, with rational values of t. Thereby a natural extension of the well-known results of G.H. Hardy and J. Gerver is obtained (Gerver established that the derivative of the function R still does exist and equals -1 at each rational point of the type $t=\frac{a}{q}$ where both numbers a and q are odd). A basic role is played by a representation of the differences of the function ψ via Poisson's summation formula and the oscillatory Fresnel integral. It is also proved that the number $\frac{3}{4}$ is the sharp value of the Lipschitz–Hölder exponent of the function ψ in the variable t almost everywhere on $\mathbb{R}^2$.

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Many papers by various authors have been devoted to Riemann's "nondifferentiable" function

$$R := \frac{2}{\pi}\sum_{n\in\mathbb{N}}\frac{\sin\pi n^2 t}{n^2}, \qquad t\in\mathbb{R},$$

during a considerable period of time.

According to K. Weierstrass [1], R was suggested by Riemann as a *plausible* example of a continuous and nowhere differentiable function. This was reported by Weierstrass on July 18, 1872, in his address before Imperial Academy of Sciences in Berlin. The main question whether Riemann was right in this presumption remained open until 1916 when G.H. Hardy [2] established that Riemann was *"almost right"*: the derivative R' indeed does not exist at any irrational point t, nor at any rational point of the kind $t=\frac{a}{q}$, where the numerator and the denominator of the (reduced) fraction are integers of different parity, i.e., $a\in\mathbb{Z}$, $q\in\mathbb{N}$, $(a,q)=1$ and $aq\in 2\mathbb{Z}$.

What can be claimed concerning the derivative in the residual rational points remained unclear until the work of J. Gerver [3] in 1970. It turned out that at each rational point $\frac{a}{q}$ where both the numerator and the denominator are odd numbers, i.e., $a\in\mathbb{Z}$, $q\in\mathbb{N}$, $(a,q)=1$, $aq\in 2\mathbb{Z}+1$, the derivative R' *still does exist* and equals -1 ! A series of papers ensued, by various authors (see, e.g., [4–15]), devoted to this function of Riemann, its versions and generalizations, in particular, to simplifications and modifications of the above remarkable result of J. Gerver.

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Here we study the function of two real variables

$$\psi := \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{\pi i(tn^2+2xn)}}{\pi i n^2}, \qquad \{t,x\}\in\mathbb{R}^2.$$

The relation between ψ and R is quite apparent: $R = \operatorname{Re}\psi|_{x=0}$; i.e., R coincides with the real part of the restriction of the ψ-function onto the t-axis.

Along with this, the ψ-function is a (generalized) solution of the Cauchy initial value problem for the Schrödinger equation

$$\left(\partial_t - \frac{\partial_x^2}{4\pi i}\right)\psi = 0, \qquad \psi = \psi[f](t,x), \qquad \psi|_{t=0} = f(x), \qquad \{t,x\}\in\mathbb{R}^2, \tag{1}$$

where ∂_t, ∂_x, and ∂_x^2 denote the operators of differentiation $\partial_t := \frac{\partial}{\partial t}$, $\partial_x := \frac{\partial}{\partial x}$, and $\partial_x^2 := \frac{\partial^2}{\partial x^2}$, and the initial function is given by

$$f = \psi|_{t=0} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi i x n}}{\pi i n^2} = 2\pi i\left(\langle x\rangle - \langle x\rangle^2 - \frac{1}{6}\right)$$

($\langle x\rangle$ is the fractional part of $x\in\mathbb{R}$).

Self-similarity is quite a natural property of solutions to problem (1) with general *periodic* initial data f, as well as to a rather broad class of Schrödinger-type equations. These solutions generate an extensive set of *functional fractals and multi-fractals*, and our concrete ψ-function is only a particular case of a *general I.M. Vinogradov's series*

$$f \sim \sum_{n\in\mathbb{Z}} \widehat{f}_n e^{2\pi i n x}, \quad x\in\mathbb{R} \quad \mapsto \quad V[f] :\sim \sum_{n\in\mathbb{Z}} \widehat{f}_n e^{2\pi i(x_1 n + x_2 n^2 + \ldots + x_r n^r)}, \quad \mathbf{x} = (x_1,\ldots,x_r)\in\mathbb{R}^r. \tag{2}$$

Some review of Vinogradov's series and integrals can be found in [16–18]; relations with the quantum optics effect of Talbot are studied in [19]. Self-similarity and fractal properties of Riemann's function R were discussed by J. Duistermaat [7]; see also the concluding remarks at the end of the present paper.

Let us introduce some definitions and state the main result of the present paper.

Let $\mathbb{Q}$ be the set of rational points on the real line $\mathbb{R}$. For a fixed $t\in\mathbb{R}$, $\mathbb{L}_t := \{t,x\}\subset\mathbb{R}^2$, $x\in\mathbb{R}$. For a fixed number $r\in\mathbb{Q}$, $r = \frac{a}{q}$, $a\in\mathbb{Z}$, $q\in\mathbb{N}$, $(a,q)=1$, and $m\in\mathbb{Z}$, let $x_m = x_m(r) := \frac{m+\alpha}{q}$ where $\alpha = 0$ if aq is even and $\alpha = \frac{1}{2}$ if aq is odd; $\mathbf{r}_m(r) := \{r, x_m\}\in\mathbb{R}^2$; $\mathbb{I}_m(r) := (\mathbf{r}_m, \mathbf{r}_{m+1})$, i.e., the open interval in $\mathbb{R}^2$ between $\mathbf{r}_m$ and $\mathbf{r}_{m+1}$;

$$\mathbb{Q}^2 := \bigcup_{r\in\mathbb{Q}}\bigcup_{m\in\mathbb{Z}} \{\mathbf{r}_m(r)\}, \qquad \mathbb{A} := \bigcup_{r\in\mathbb{Q}} \mathbb{L}_r, \qquad \mathbb{B} := \mathbb{A}\setminus\mathbb{Q}^2 = \bigcup_{r\in\mathbb{Q}}\bigcup_{m\in\mathbb{Z}} \mathbb{I}_m(r). \tag{3}$$

Theorem 1. 1. *The partial derivative $\partial_x\psi$ of the function ψ with respect to the variable x exists everywhere on $\mathbb{R}^2\setminus\mathbb{Q}^2$ and is bounded on this set*; *in particular, $\partial_x\psi\in\mathcal{L}^\infty(\mathbb{R}^2)$.*

2. *The partial derivatives $\partial_t\psi$ and $\partial_x^2\psi/4\pi i$ exist everywhere on $\mathbb{B}$. Both of them equal -1 on this set, so that everywhere on $\mathbb{B}$ the function ψ satisfies the Schrödinger equation* (1) *in the classical sense. None of these derivatives exist at any point of $\mathbb{Q}^2$.*

3. *The local Lipschitz–Hölder smoothness exponent of ψ in the variable t equals $\frac{3}{4}$ almost everywhere on $\mathbb{R}^2$*: *for each $\varepsilon > 0$ at almost every point $\{t,x\}\in\mathbb{R}^2$ the following relations are true*:

$$\limsup_{\tau\to 0} \frac{|\psi(t+\tau,x)-\psi(t,x)|}{|\tau|^{3/4}|\log|\tau||^{2+\varepsilon}} = 0, \qquad \limsup_{\tau\to 0} \frac{|\psi(t+\tau,x)-\psi(t,x)|}{|\tau|^{3/4}} = \infty. \tag{4}$$

In particular, the partial derivative $\partial_t\psi$ does not exist at almost any point on $\mathbb{R}^2$.

The aforementioned result of Gerver is incorporated in this theorem. Indeed, the points $\{t,x\} = \{r,0\}$, $r=\frac{a}{q}$, $a\in\mathbb{Z}$, $q\in\mathbb{N}$, $(a,q)=1$, for $aq\in 2\mathbb{Z}+1$ belong to the set $\mathbb{B}$: they are mid-points of the intervals $\mathbb{I}_m(r)$. On the other hand, if $aq\in 2\mathbb{Z}$, then the points $\{r,0\}$ are already in $\mathbb{Q}^2$: they are end-points of $\mathbb{I}_m(r)$, and on $\mathbb{Q}^2$ the derivative $\partial_t\psi$ does not exist.

Let us also note that the role of the number $\frac{3}{4}$ as the local smoothness exponent of the function R was pinpointed in Hardy's work [2], as well as in subsequent publications (see, e.g., [7]).

We will provide some more detailed comments at the end of this paper, after the proof of our theorem.

Proof of Theorem 1. The formal (termwise) application of the differential operator to the series that defines our ψ-function generates the series

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\partial_x}{4\pi i}\left(\frac{e^{\pi i(tn^2+2xn)}}{\pi i n^2}\right) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{\pi i(tn^2+2xn)}}{2\pi i n}.$$

The symmetric partial sums of the series on the right-hand side,

$$H_N := \sum_{0<|n|\le N} \frac{e^{\pi i(tn^2+2xn)}}{2\pi i n}, \qquad N\in\mathbb{N},$$

are discrete Hilbert transformations with the quadratic phase. They constitute a globally bounded and everywhere convergent sequence:

$$\sup_{N\in\mathbb{N},\,\{t,x\}\in\mathbb{R}^2} |H_N| < \infty, \qquad \forall\{t,x\}\in\mathbb{R}^2 \quad \exists \lim_{N\to\infty} H_N := H.$$

The latter facts are a particular case of a more general statement on bounds and convergence of sequences of discrete Hilbert transformations with a real algebraic polynomial phase which was established in the paper by G.I. Arkhipov and K.I. Oskolkov [20].[1] In addition, the restriction $H|_{\mathbb{L}_t}$ on each line $\mathbb{L}_t$ parallel to the x-axis is a function of bounded *weak quadratic* variation in the variable x on the period $(0,1]$, and this is true uniformly in $t\in\mathbb{R}$ (see [21]). Further, if t is an irrational number, then $H|_{\mathbb{L}_t}$ is a continuous function of x, and if it is rational, $t=r\in\mathbb{Q}$, then $x+H|_{\mathbb{L}_r}$ is piecewise constant, with jumps at the points $x=x_m$. Thus, it follows that the sequence $\{H_N\}$ converges uniformly in x on each line $\mathbb{L}_t$ with an irrational t. On the other hand, if $t=r\in\mathbb{Q}$, then the convergence of $\{H_N|_{\mathbb{L}_r}\}$ is uniform in x on compact subintervals of (x_m, x_{m+1}), $m\in\mathbb{Z}$, between the neighboring jump-points of H on the line $\mathbb{L}_r$. This completes the proof of claim 1 of our theorem.

For the proof of claim 2, we will need some auxiliary statements.

Lemma 1. *The following identity is valid*:

$$\int_{\mathbb{R}} \frac{e^{\pi i t y^2}-1}{\pi i y^2} e^{2\pi i y x}\,dy = \sqrt{|t|}\begin{cases} J\left(\dfrac{x}{\sqrt{t}}\right), & t>0,\\ -J^*\left(\dfrac{x}{\sqrt{|t|}}\right), & t<0, \end{cases} \tag{5}$$

where J^* denotes the complex conjugate of J,

$$J := 2\sqrt{i}\,|x|\int_{|x|}^{\infty} \frac{e^{-\pi i y^2}}{y^2}\,dy, \qquad x\in\mathbb{R}, \quad \sqrt{i} := e^{\frac{\pi i}{4}} = \frac{1+i}{\sqrt{2}}.$$

[1] Our particular case of the quadratic phase can be proved using simpler tools, without addressing the results of the method of exponential sums by I.M. Vinogradov for H. Weil's sums.

Further,

$$\text{(i)}\quad J \equiv 2\sqrt{i}\left(e^{-\pi i x^2} - 2\pi i|x| \int\limits_{|x|}^{\infty} e^{-\pi i y^2}\,dy\right), \qquad \text{(ii)}\quad J = O\big(|x|^{-2}\big), \quad |x| \to \infty, \tag{6}$$

and[2]

$$t + \psi(t,x) - \psi(0,x) = \sum_{n\in\mathbb{Z}} \frac{e^{\pi i t n^2} - 1}{\pi i n^2} e^{2\pi i n x} = \sqrt{t} \sum_{m\in\mathbb{Z}} J\left(\frac{x+m}{\sqrt{t}}\right). \tag{7}$$

In particular, our ψ-function is differentiable in the variable t everywhere on the line $t = 0$ except for the integral points x, and $\partial_t\psi|_{t=0} = -1$, $x \in \mathbb{R}\setminus\mathbb{Z}$.

Proof. Let $(\mathrm{R})\int_{\mathbb{R}}\dots$ denote the improper integral in the sense of Riemann on the real line. The following "complete Fresnel's integral" is well-known:

$$(\mathrm{R})\int\limits_{\mathbb{R}} e^{\pi i y^2}\,dy = \sqrt{i},$$

which implies the following well-known representation of the Green's function Γ of problem (1):

$$\Gamma := (\mathrm{R})\int\limits_{\mathbb{R}} e^{\pi i(ty^2 + 2xy)}\,dy = \sqrt{\frac{i}{t}}\, e^{-\frac{\pi i x^2}{t}}, \quad t > 0, \qquad \Gamma(t,x) = \Gamma^*(|t|,x), \quad t < 0, \tag{8}$$

whereas the integral converges uniformly in $\{t,x\}$ on compact subsets of $\mathbb{R}^2$ outside the line $t = 0$. First assume that $t > 0$ and $x \neq 0$. Let us integrate both sides of relation (8), reverse the order of integration in the repeated integral, and then introduce the substitution of the variable of integration which is indicated over the equality sign:

$$\int\limits_0^t \Gamma\,d\tau = \int\limits_0^t \left((\mathrm{R})\int\limits_{\mathbb{R}} e^{\pi i(\tau y^2 + 2xy)}\,dy\right) d\tau = (\mathrm{R})\int\limits_{\mathbb{R}} \left(\int\limits_0^t e^{\pi i(\tau y^2 + 2xy)}\,d\tau\right) dy$$

$$= \int\limits_{\mathbb{R}} \frac{e^{\pi i t y^2} - 1}{\pi i y^2} e^{2\pi i y x}\,dy = \int\limits_0^t \sqrt{\frac{i}{\tau}}\, e^{-\frac{\pi i x^2}{\tau}}\,d\tau \overset{\tau = \frac{x^2}{y^2}}{=} 2\sqrt{i}\,|x| \int\limits_{|x|/\sqrt{t}}^{\infty} \frac{e^{-\pi i y^2}}{y^2}\,dy.$$

This and the definition of the function J imply (5).

Relations (6) follow by integrations by parts in "mutually opposite directions" (it suffices to consider only positive x):

$$x\int\limits_x^{\infty} \frac{e^{-\pi i y^2}}{y^2}\,dy = -x\frac{e^{-\pi i y^2}}{y}\bigg|_{y=x}^{\infty} - x\int\limits_x^{\infty} \frac{2\pi i y e^{-\pi i y^2}}{y}\,dy = e^{-\pi i x^2} - 2\pi i x\,(\mathrm{R})\int\limits_x^{\infty} e^{-\pi i y^2}\,dy,$$

which proves (i) in (6), and further

$$J = \sqrt{i}x\int\limits_x^{\infty} \frac{d(e^{-\pi i y^2})}{-\pi i y^3}\,dy = \sqrt{i}x\left(\frac{e^{-\pi i y^2}}{-\pi i y^3}\bigg|_{y=x}^{\infty} - 3\int\limits_x^{\infty} \frac{e^{-\pi i y^2}}{\pi i y^4}\,dy\right) = \frac{1}{\sqrt{i}}\left(\frac{e^{-\pi i x^2}}{\pi x^2} - 3x\int\limits_x^{\infty} \frac{e^{-\pi i y^2}}{\pi y^4}\,dy\right),$$

[2] For brevity, in the sequel we will make use of "somewhat abusive" notation $\sqrt{t}\,J\big(\frac{x}{\sqrt{t}}\big)$ for negative t, meaning $-\sqrt{|t|}J^*\Big(\frac{x}{\sqrt{|t|}}\Big)$ in this case.

which implies (ii) in (6) as well as a more exact asymptotic formula

$$J = \frac{e^{-\pi i x^2}}{\pi\sqrt{i}\,x^2} + O\biggl(\frac{1}{x^4}\biggr), \qquad x \to \infty. \tag{9}$$

The first of relations (7) is a direct corollary of the definition of our ψ-function. To prove the second equality in (7), for a fixed $t \neq 0$, let us consider (5) as the representation of the function $F := \sqrt{t}J\bigl(\frac{x}{\sqrt{t}}\bigr)$ of the variable $x \in \mathbb{R}$ by the Fourier integral:

$$F = \int\limits_{\mathbb{R}} \widehat{F}(y)e^{2\pi i y x}\,dy, \qquad \frac{e^{\pi i t y^2}-1}{\pi i y^2} = \widehat{F}(y) = \int\limits_{\mathbb{R}} F(x)e^{-2\pi i x y}\,dx.$$

Keeping in mind estimate (ii) in (6), we see that F and $\widehat{F}$ are represented by absolutely convergent Fourier integrals, and therefore Poisson's summation formula is valid (see, e.g., [22, Ch. 2]):

$$\sum_{m\in\mathbb{Z}} \sqrt{t}J\biggl(\frac{x+m}{\sqrt{t}}\biggr) = \sum_{m\in\mathbb{Z}} F(x+m) = \sum_{n\in\mathbb{Z}} \widehat{F}(n)e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} \frac{e^{\pi i t n^2}-1}{\pi i n^2}e^{2\pi i n x}.$$

This proves identity (7).

Making use of this identity, as well as estimate (ii) in (6), we see that if $|x| \le \frac{1}{2}$, then

$$1 + \frac{\psi(t,x)-\psi(0,x)}{t} = \frac{1}{\sqrt{t}}\Biggl(J\biggl(\frac{x}{\sqrt{t}}\biggr) + \sum_{m\in\mathbb{Z}\setminus\{0\}} O\biggl(\frac{|t|}{m^2}\biggr)\Biggr) = \frac{1}{\sqrt{t}}J\biggl(\frac{x}{\sqrt{t}}\biggr) + O\bigl(\sqrt{|t|}\bigr). \tag{10}$$

If $x \neq 0$, then, again making use of (ii) in (6), we derive

$$\frac{\psi(t,x)-\psi(0,x)}{t} = -1 + O\biggl(\frac{\sqrt{|t|}}{x^2}\biggr) \to -1, \qquad t \to 0,$$

and this implies that the derivative $\partial_t\psi$ exists and equals -1 everywhere for $0 < |x| \le \frac{1}{2}$, and therefore everywhere on the x-axis except for the integral points, because ψ is periodic, of period 1. That the derivative does not exist at the integral points is also a corollary of (10), because $J(0) = 2\sqrt{i} \neq 0$. The proof of the lemma is complete.

Now we address the ψ-functions in a neighborhood of a fixed line $\mathbb{L}_r$, $r \in \mathbb{Q}$. We have

$$\begin{gathered} \Psi(t+r,x) = \sum_{n\in\mathbb{Z}} e^{\pi i r n^2}\widehat{f}_n e^{\pi i t n^2}e^{2\pi i n x}, \\ t + \psi(t+r,x) - \psi(r,x) = \sum_{n\in\mathbb{Z}} e^{\pi i r n^2}\frac{e^{\pi i t n^2}-1}{\pi i n^2}e^{2\pi i n x}. \end{gathered} \tag{11}$$

Let us consider these representations as Fourier series in the variable x for a fixed t. Then we see that the right-hand sides are multiplier transformations of the original Fourier series

$$\Psi(t,x) = \sum_{n\in\mathbb{Z}} \widehat{f}_n e^{\pi i t n^2}e^{2\pi i n x}, \qquad t + \psi(t,x) - \psi(0,x) = \sum_{n\in\mathbb{Z}} \frac{e^{\pi i t n^2}-1}{\pi i n^2}e^{2\pi i n x}.$$

Since $r \in \mathbb{Q}$, the multiplier $\mu := \{\mu_n := e^{\pi i r n^2}\}_{n\in\mathbb{Z}}$ is periodic on $\mathbb{Z}$. More explicitly, if $r = \frac{a}{q}$, $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q) = 1$, then $\mu_{n+2q} = \mu_n$, $n \in \mathbb{Z}$, and in still more detail, $\mu_{n+q} = (-1)^{aq}\mu_n$, $n \in \mathbb{Z}$; i.e., the number q is a period of μ if $aq \in 2\mathbb{Z}$ and is its “antiperiod” if $aq \in 2\mathbb{Z}+1$.

Lemma 2. *Let $q \in \mathbb{N}$ and $\mu := \{\mu_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers whose "quasi-period" equals q, which means that there exists a number $\alpha \in \mathbb{R}$ such that $\mu_{n+q} = e^{2\pi i\alpha}\mu_n$, $n \in \mathbb{Z}$; let $M = M_\mu$ be the multiplier transformation of Fourier series generated by μ, i.e., $M\colon f \mapsto M[f]$, where $\widehat{M_n}[f] = \mu_n \widehat{f}$, and let*

$$x_m := \frac{m+\alpha}{q}, \qquad \widehat{\mu}_m := \frac{1}{q}\sum_{n=1}^{q}\mu_n e^{-2\pi i x_m n}, \qquad m \in \mathbb{Z}.$$

Then

$$M[f](x) = \sum_{m=1}^{q}\widehat{\mu}_m f(x+x_m), \tag{12}$$

and if $f = \sum_{m\in\mathbb{Z}} F(x+m)$, i.e., f results from periodization[3] *of a function F on the real line $\mathbb{R}$, then*

$$M[f](x) = \sum_{m\in\mathbb{Z}}\widehat{\mu}_m F(x+x_m). \tag{13}$$

In particular (see (3), (11)),

$$\begin{aligned}
&\text{(i)}\ \ \Psi(t+r,x) = \sum_{m=1}^{q}\widehat{S}(\mathbf{r}_m)\Psi(t,x+x_m), \qquad \textit{where}\ \ \widehat{S}(\mathbf{r}_m) := \frac{1}{q}\sum_{n=1}^{q}e^{\pi i(rn^2-2x_m n)},\\
&\text{(ii)}\ \ t+\psi(t+r,x)-\psi(r,x) = \sqrt{t}\sum_{m\in\mathbb{Z}}\widehat{S}(\mathbf{r}_m)J\Big(\frac{x+x_m}{\sqrt{t}}\Big),
\end{aligned} \tag{14}$$

and we have

$$\text{(i)}\ \ \big|\widehat{S}(\mathbf{r}_m)\big| = q^{-1/2}, \quad m \in \mathbb{Z}; \qquad \text{(ii)}\ \ \sum_{m=1}^{q}\widehat{S}(\mathbf{r}_m) = 1. \tag{15}$$

Proof. It is clear that it suffices to establish (12) only for the exponents $e^{2\pi i N x}$, $N \in \mathbb{Z}$, i.e., one needs to prove that for each exponent $f := e^{2\pi i N x}$ with an integral N the sum on the right-hand side of (12) equals $\mu_N f$. Let us represent N as $N = \nu + Mq$, where $\nu \in [1,q]$ and $M \in \mathbb{Z}$. Then, by quasi-periodicity, $\mu_N = \mu_\nu e^{2\pi i M\alpha}$, and one has

$$\begin{aligned}
\sum_{m=1}^{q}\widehat{\mu}_m f(\cdot + x_m) &= f\sum_{m=1}^{q}\widehat{\mu}_m e^{2\pi i N x_m} = \frac{f}{q}\sum_{n=1}^{q}\mu_n\sum_{m=1}^{q}e^{2\pi i(N-n)x_m}\\
&= f\sum_{n=1}^{q}\mu_n e^{\frac{2\pi i(N-n)\alpha}{q}}\Bigg(\frac{1}{q}\sum_{m=1}^{q}e^{\frac{2\pi i(N-n)m}{q}}\Bigg) = f\mu_\nu e^{2\pi i M\alpha} = \mu_N f,
\end{aligned}$$

which is precisely what we needed. Relation (13) follows from (12) if we take into account that $\widehat{\mu}_{m+q} = \widehat{\mu}_m$, $m \in \mathbb{Z}$, and (14) are particular cases of (12) and (13) in view of (7).

For the sake of completeness, let us also prove (i) in (15), although the argument is not much different from the routine evaluation of the complex moduli of the classical Gauss' sums. Note that in contrast with the sequence $\{\mu_n := e^{\pi i r n^2}\}_{n\in\mathbb{Z}}$ for which q in the case $aq \in 2\mathbb{Z}+1$ is only a quasi-period, for each fixed $m \in \mathbb{Z}$ the sequence $\{\mu_{m,n} := e^{\pi i(rn^2-2x_m n)}\}_{n\in\mathbb{Z}}$ has a true period q, i.e.,

[3] We assume that the series converges absolutely.

$\mu_{m,n+q} = \mu_{m,n}$, $n \in \mathbb{Z}$. Therefore, making use of the fact that in our case $\mu_{m,n+\nu}\mu^*_{m,\nu} = \mu_{m,n}e^{2\pi irn\nu}$, $n, \nu \in \mathbb{Z}$, $\mu_{m,q} = 1$, and applying the well-known trick of "shifting the summation interval," we obtain

$$\left|\sum_{n=1}^{q}\mu_{m,n}\right|^2 = \left(\sum_{n=1}^{q}\mu_{m,n}\right)\left(\sum_{\nu=1}^{q}\mu^*_{m,\nu}\right) = \sum_{\nu=1}^{q}\left(\sum_{n=1}^{q}\mu_{m,n+\nu}\mu^*_{m,\nu}\right) = \sum_{n=1}^{q}\mu_{m,n}\sum_{\nu=1}^{q}e^{2\pi irn\nu} = q,$$

since on the interval of the outside summation, the sum inside is different from 0 only if $an \equiv 0 \pmod q$, which occurs only for $n = q$. Equality (ii) in (15) is a corollary of (i) in (14) if we notice that $\Psi := 1$ is a solution to problem (1) with the initial data $f := 1$ (a propos, (ii) in (15) can be easily deducted directly from the definition of $\widehat{S}$). This completes the proof of the lemma.

Now we finish the proof of claim 2 of Theorem 1. As we already mentioned above,

$$\frac{\partial_x\psi}{4\pi i} = (\text{p.v.})\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{e^{\pi i(tn^2+2xn)}}{2\pi in} = H(t,x), \qquad \{t,x\} \in \mathbb{R}^2,$$

and H is continuous on $\mathbb{R}^2 \setminus \mathbb{Q}^2$. The function H is a generalized solution of (1) with the initial data

$$f = H|_{\mathbb{L}_0} = (\text{p.v.})\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{e^{2\pi ixn}}{2\pi in} = \sum_{n\in\mathbb{N}}\frac{\sin 2\pi xn}{\pi n} = \frac{1}{2} - \langle x\rangle.$$

Clearly $\partial_x f = -1$, $x \in \mathbb{R} \setminus \mathbb{Z}$, and at each point of $\mathbb{Z}$ this initial function experiences a jump of 1. Therefore, using (i) in (14), the last claim of Lemma 1 and (ii) in (15), we infer that everywhere on the line $\mathbb{L}_r$, except for the points $\mathbf{r}_m$,

$$\left.\frac{\partial_x^2\psi}{4\pi i}\right|_{\mathbb{L}_r} = \sum_{m=1}^{q}\widehat{S}(\mathbf{r}_m)\partial_x H(0, x + x_m) = \sum_{m=1}^{q}\widehat{S}(\mathbf{r}_m)(-1) = -1,$$

$$\partial_t\psi|_{\mathbb{L}_r} = \sum_{m=1}^{q}\widehat{S}(\mathbf{r}_m)\partial_t\psi(\cdot, x + x_m)\Big|_{\mathbb{L}_0} = \sum_{m=1}^{q}\widehat{S}(\mathbf{r}_m)(-1) = -1,$$

and that at none of the points $\mathbf{r}_M$, $M = 1, \dots, q$, the derivatives exist, because $\widehat{S}(\mathbf{r}_M) \neq 0$ in accordance with (i) in (15).

Now we prove claim 3. The following lemma provides an analog of estimate (10), localized to a neighborhood of a point $\mathbf{r} \in \mathbb{Q}^2$.

Lemma 3. *Let* (*see* (3)) $|x + x_M| \le \frac{1}{2q}$. *Then*

$$\tau + \psi(r + \tau, x) - \psi(r, x) = \widehat{S}(\mathbf{r}_M)\sqrt{\tau}\, J\left(\frac{x + x_M}{\sqrt{\tau}}\right) + O\bigl(|q\tau|^{3/2}\bigr). \tag{16}$$

Proof. Estimate (16) follows from (14), (i) in (15), and (ii) in (6), because

$$\sum_{m\neq M}\left|\widehat{S}(\mathbf{r}_m)\sqrt{\tau}\, J\left(\frac{x + x_m}{\sqrt{\tau}}\right)\right| = \sum_{m\neq 0}O\bigl(q^{-1/2}|\tau|^{1/2}|\tau|q^2m^{-2}\bigr) = O\bigl(|q\tau|^{3/2}\bigr).$$

Let us fix an irrational $t \in (0, 1)$ and consider the sequence $r_j = \frac{a_j}{q_j} \in \mathbb{Q}$, $j = 1, 2, \dots$, of *convergent fractions* for t. This sequence possesses the following properties (see [23]):

$$\text{(i)}\ \frac{1}{2q_jq_{j+1}} \le |t - r_j| \le \frac{1}{q_jq_{j+1}}, \qquad \text{(ii)}\ \frac{a_j}{q_j} - \frac{a_{j+1}}{q_{j+1}} = \frac{(-1)^j}{q_jq_{j+1}}, \qquad \text{(iii)}\ q_{j+1} = k_jq_j + q_{j-1}, \tag{17}$$

where $k_j \in \mathbb{N}$, $j = 1, 2, \dots$, is the sequence of *partial quotients* of t. Let us find, for a given x, a sequence $M_j \in \mathbb{N}$, $j = 1, 2, \dots$, such that $-1 \le 2q_j(x + x_{M_j}) < 1$ (it is easy to see that this sequence is unique), and let

$$\eta_j := t - r_j, \qquad \tau_j := t + \tau - r_j = \eta_j + \tau, \qquad \xi_j = \xi_j(t,x) := q_j(x + x_{M_j}). \tag{18}$$

Then, with the help of (16), (ii) in (6), (i) in (15), and (i) in (17), we obtain

$$\begin{aligned} \eta_j + \psi(t,x) - \psi(r_j,x) &= \widehat{S}(\mathbf{r}_{M_j})\sqrt{\eta_j}\, J\biggl(\frac{\xi_j}{q_j\sqrt{\eta_j}}\biggr) + O\bigl(|q_j\eta_j|^{3/2}\bigr) = O\bigl(|\xi_j|^{-2}q_{j+1}^{-3/2}\bigr), \\ \tau_j + \psi(t+\tau,x) - \psi(r_j,x) &= \widehat{S}(\mathbf{r}_{M_j})\sqrt{\tau_j}\, J\biggl(\frac{\xi_j}{q_j\sqrt{\tau_j}}\biggr) + O\bigl(|q_j\tau_j|^{3/2}\bigr) = O\bigl(|\xi_j|^{-2}|q_j\tau_j|^{3/2}\bigr) \\ &= O\bigl(|\xi_j|^{-2}|q_j\eta_j|^{3/2}\bigr) + O\bigl(|\xi_j|^{-2}|q_j\tau|^{3/2}\bigr) = O\bigl(|\xi_j|^{-2}q_{j+1}^{-3/2}\bigr) + O\bigl(|\xi_j|^{-2}|q_j\tau|^{3/2}\bigr). \end{aligned} \tag{19}$$

Now assume that $\frac{1}{q_{j+1}^2} \le |\tau| \le \frac{1}{q_j^2}$. Then obviously $\max\bigl\{q_{j+1}^{-3/2}, |q_j\tau|^{3/2}\bigr\} \le |\tau|^{3/4}$, and (19) implies the following estimate uniform in $j \in \mathbb{N}$ and $x \in \mathbb{R}$:

$$|\psi(t+\tau,x) - \psi(t,x)| = O_x\bigl(|\xi_j|^{-2}|\tau|^{3/4}\bigr) \qquad \text{for} \quad \frac{1}{q_{j+1}^2} \le |\tau| \le \frac{1}{q_j^2}, \tag{20}$$

where the notation O_x means that the corresponding constant in O depends on x.

Consider an arbitrary sequence of positive numbers $\{\delta_j\}_1^\infty$ such that $\sum_1^\infty \delta_j < \infty$, and define the following point sets on $\mathbb{R}$ (see (18)):

$$\mathbb{E} = \mathbb{E}_{t,\delta} := \limsup_{j\to\infty}\bigl\{x\colon |\xi_j(t,x)| \le \delta_j\bigr\} = \bigcap_{k\in\mathbb{N}}\bigcup_{j\ge k}\bigl\{x\colon |\xi_j(t,x)| \le \delta_j\bigr\},$$

$$\mathbb{G} = \mathbb{G}_{t,\delta} := \mathbb{R}\setminus\mathbb{E} = \bigcup_{k\in\mathbb{N}}\bigcap_{j\ge k}\bigl\{x\colon |\xi_j(t,x)| > \delta_j\bigr\}.$$

These sets are periodic, of period 1, and as is easily seen from definition (18), $\operatorname{meas}\{x \in [0,1)\colon |\xi_j(t,x)| \le \delta_j\} = \delta_j$. Since $\sum_1^\infty \delta_j < \infty$, we have $\operatorname{meas}\mathbb{E} = 0$; that is, almost every $x \in \mathbb{R}$ belongs to $\mathbb{G}$, while (20) implies that

$$|\psi(t+\tau,x) - \psi(t,x)| = O_x\bigl(|\delta_j|^{-2}|\tau|^{3/4}\bigr) \qquad \text{for} \quad \frac{1}{q_{j+1}^2} \le |\tau| \le \frac{1}{q_j^2}, \quad x \in \mathbb{G}. \tag{21}$$

Let us consider a concrete sequence $\delta = \{\delta_j\}$; for example, let $\delta_j := \varphi(\log q_j)$, $j \in \mathbb{N}$, where φ, $x \in \mathbb{R}_+$, is a monotonically decreasing function on the positive semi-axis $\mathbb{R}_+$, with $\sum_1^\infty \varphi(q) < \infty$. Since the sequence $\{q_j\}$ grows no slower than the sequence of Fibonacci numbers (see (iii) in (17)), i.e., at least as a geometric progression, we easily infer that $j = O(\log q_j)$ and $\sum_1^\infty \delta_j < \infty$. In addition, it is obvious that $\delta_j \ge \varphi(|\log|\tau||)$ for $|\tau| \le q_j^{-2}$. Thus, (21) implies the following strengthened version of the first of relations (4).

Lemma 4. *Assume that φ is a monotonically decreasing positive function on the positive semi-axis $\mathbb{R}_+$ with $\sum_1^\infty \varphi(q) < \infty$. Then, for every irrational t and almost every $x \in \mathbb{R}$,*

$$\limsup_{\tau\to 0}\frac{\varphi^2(|\log|\tau||)|\psi(t+\tau,x) - \psi(t,x)|}{|\tau|^{3/4}} = 0, \tag{22}$$

and in particular, for every $\varepsilon > 0$,

$$\limsup_{\tau\to 0}\frac{|\psi(t+\tau,x) - \psi(t,x)|}{|\tau|^{3/4}|\log|\tau||^2(\log|\log|\tau||)^{2+\varepsilon}} = 0.$$

Now we prove the second of relations (4). We start from the lower estimate for $|J|$:

$$J \gg \frac{1}{1+x^2}, \quad x \in \mathbb{R} \tag{23}$$

($\gg$ is Vinogradov's symbol: the relation $f \gg g$ means that $|g| \le c|f|$ everywhere in the domain of the complex-valued functions f and g, with c a finite positive constant). This estimate is easily implied by the asymptotic formula (9) if we additionally notice that $J \ne 0$, $x \in \mathbb{R}$.

With the help of (23) (see (16)–(19)) we infer

$$\begin{aligned}|\psi(t,x) - \psi(r_j,x)| &= |\psi(r_j+\eta_j,x) - \psi(r_j,x)| \gg q_j^{-1/2}(q_jq_{j+1})^{-1/2}\frac{q_j}{q_j+\xi_j^2q_{j+1}} + O\big(q_{j+1}^{-3/2}\big)\\ &= q_{j+1}^{-1/2}\big(q_j+\xi_j^2q_{j+1}\big)^{-1} + O\big(q_{j+1}^{-3/2}\big).\end{aligned}$$

This, together with (i) and (iii) in (17), implies that

$$|\psi(t,x) - \psi(r_j,x)| \gg q_{j+1}^{-1/2}q_j^{-1} + O\big(q_{j+1}^{-3/2}\big) \gg |t-r_j|^{3/4}k_j^{1/4}\big(1+O(k_j^{-1})\big) \quad \text{if} \quad |\xi_j| \le k_j^{-1/2}. \tag{24}$$

Given a number $t \in \mathbb{R}$ and a fixed (large) number K, let us denote

$$\mathbb{J}_K(t) := \big\{j \in \mathbb{N}\colon\ k_j(t) \ge K\big\}, \qquad \mathbb{T}_K := \left\{t \in \mathbb{R}\colon \sum_{j\in\mathbb{J}_K(t)} k_j^{-1/2}(t) = \infty\right\}. \tag{25}$$

If $t \in \mathbb{R}\setminus\mathbb{T}_K$ then $\sum_{j\in\mathbb{J}_K(t)} k_j^{-1/2}(t) < \infty$, and consequently the set $\mathbb{J}_K(t)$ is either finite (possibly, empty) or infinite, but then for the sequence $k_j(t)$ the estimate $k_j(t) = O(j^2)$, $j \to \infty$, is *not true*. The latter could occur only on the set of points t of measure 0, in view of the following well-known result due to A.Ya. Khinchin [23, Theorem 30].

Theorem 2 (A.Ya. Khinchin). *Let $K(j)$ be an arbitrary positive function of natural argument j. The inequality $k_j = k_j(t) \ge K(j)$ for almost all t is true for an infinite number of j's if the series formed by the numbers $1/K(j)$ is divergent. On the contrary, the inequality for almost all t is true only for a finite number of j's if the series formed by the numbers $1/K(j)$ is convergent.*

Thus, the set $\mathbb{T}_K$ is of full measure. Now assume that K is sufficiently large and $t \in \mathbb{T}_K$. Then for almost all x

$$\limsup_{j\in\mathbb{J}_K(t)} \frac{|\psi(t,x) - \psi(r_j,x)|}{|t-r_j|^{3/4}} \gg K^{1/4}. \tag{26}$$

Indeed, it follows from (24) that relation (26) is true at each point $x \in \mathbb{X}(t)$ where

$$\mathbb{X}(t) := \limsup_{j\in\mathbb{J}_K(t),\ j\to\infty} \mathbb{X}_j(t), \qquad \mathbb{X}_j(t) := \big\{x\colon\ |\xi_j| \le k_j^{-1/2}(t)\big\},$$

and $\operatorname{meas}\mathbb{R}\setminus\mathbb{X}(t) = 0$ in view of the divergence of the series $\sum_{j\in\mathbb{J}_K(t)} k_j^{-1/2}$. The latter easily follows if we notice that $2\operatorname{meas}\mathbb{X}_j(t)\cap\mathbb{I} \ge k_j^{-1/2}(t)|\mathbb{I}|$ for every fixed interval $\mathbb{I} \subset \mathbb{R}$ and for all sufficiently large j.

Summarizing, we find that (26) is true for almost all $\{t,x\} \in \mathbb{R}^2$, and since the number K is arbitrarily large, the second of relations (4) in claim 3 of our theorem follows.

Remark 1. Relations (14) reflect a general self-similarity property of the solutions of problem (1) with periodic initial data: the Ψ-function is "self-reproductive"; it transports a certain "pattern function" into neighborhoods of rational points $\mathbb{Q}^2$ on the plane $\mathbb{R}^2$. This pattern is "scaled" by the Gauss sums $\widehat{S}(\mathbf{r})$, $\mathbf{r} \in \mathbb{Q}^2$. One can see from (ii) in (14) that the Fresnel integral J is a pattern of such kind for the local increments of our ψ-function near $\mathbb{Q}^2$.

Certainly, this is not an exceptional property exhibited only by Ψ-functions. All Vinogradov series (2) possess the self-similarity property: the role of the quadratic Gaussian sums is taken over by the complete rational exponential sums of higher degree. Patterns for Vinogradov series are oscillatory exponential integrals.

Remark 2. A partial description of the smoothness of restrictions of the function ψ onto lines parallel to the coordinate axes is expressed via the imbedding $\psi \in \mathcal{H}^{3/4}(\mathcal{L}^{2,\infty}) \cap \mathcal{H}^{3/2}(\mathcal{L}^{\infty,2})$, which in terms of the Lipschitz–Hölder exponents can be deciphered as follows:

$$\begin{aligned} &\text{(i)}\quad \max_x \left(\int_0^2 |\psi(t+\tau,x) - \psi(t,x)|^2\,dt \right)^{1/2} = O\big(|\tau|^{3/4}\big), \qquad \tau \to 0, \\ &\text{(ii)}\quad \max_t \left(\int_0^1 |\partial_x\psi(t,x+\xi) - \partial_x\psi(t,x)|^2\,dx \right)^{1/2} = O\big(|\xi|^{1/2}\big), \qquad \xi \to 0. \end{aligned} \tag{27}$$

Indeed, let us consider the finite partial sums of the series for ψ:

$$\psi_m := \sum_{0<|n|\le m} \frac{e^{\pi i(tn^2+2xn)}}{\pi i n^2}, \qquad m \in \mathbb{N}.$$

By Parseval's equality,

$$\|\psi - \psi_m;\, \mathcal{L}^2\|^2(x) := \int_0^2 |\psi - \psi_m|^2\,dt = \frac{8}{\pi^2} \sum_{n>m} \frac{\cos^2 2\pi n x}{n^4} = O(m^{-3}),$$

$$\|\psi - \psi_m;\, \mathcal{L}^2\|^2(t) := \int_0^1 |\psi - \psi_m|^2\,dx = \frac{2}{\pi^2} \sum_{n>m} \frac{1}{n^4} = O(m^{-3}), \qquad m \to \infty.$$

Since ψ_m is a trigonometric polynomial of degree m^2 in the variable t and of degree m in the variable x, estimates (27) follow from the inverse theorems of approximation theory.

In the uniform metric, a partial description of the restrictions of ψ is given by $\psi \in \mathcal{H}^{1/2,1}(\mathcal{L}^{\infty,\infty})$, which means the following:

$$\begin{aligned} &\text{(i)}\quad \max_{t,x} |\psi(t+\tau,x) - \psi(t,x)| = O\big(|\tau|^{1/2}\big), \qquad \tau \to 0, \\ &\text{(ii)}\quad \max_{t,x} |\psi(t,x+\xi) - \psi(t,x)| = O(|\xi|), \qquad \xi \to 0. \end{aligned} \tag{28}$$

Relation (i) in (28) follows, with the help of the inverse theorems of approximation theory, from the trivial estimate

$$\|\psi - \psi_m;\, \mathcal{L}^{\infty,\infty}\| = \max_{t,x} |\psi - \psi_m| \le \frac{2}{\pi} \sum_{n>m} \frac{1}{n^2} = O(m^{-1}), \qquad m \to \infty,$$

while (ii) in (28) is a corollary of the global boundedness of the partial derivative $\partial_x\psi$ (see claim 1 of Theorem 1).

Along with this, it follows from (16) that the smoothness exponent $\frac{1}{2}$ in (i) of (28) is *sharp* at all points of the set $\mathbb{Q}^2$; at all points of $\mathbb{B}$ such a sharp exponent equals 1 in view of claim 2 of our theorem. By claim 3, the number $\frac{3}{4}$, which is obviously the center of the interval $\left[\frac{1}{2}, 1\right]$, serves, in terms of the Lipschitz–Hölder classes, as a sharp smoothness exponent in the variable t at almost all points of $\mathbb{R}^2$, and simultaneously as a smoothness exponent in $\mathcal{L}^2$ (see (i) in (27)).

A natural problem arises from these observations: given a number α in the interval $\left(\frac{1}{2},1\right)$, *characterize* the set of those points on $\mathbb{R}^2$ where α is the sharp (optimal) smoothness exponent of ψ in the variable t. The traditional characterization terminology for *thin* sets is the Hausdorff dimension (see [24]). In particular, it is plausible that the Hausdorff dimension of such sets is nontrivial, which would mean that the function ψ is a *multi-fractal* in a strict sense.

It is the authors' intention to address this problem in the near future.

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