Pell's equation and side and diagonal numbers

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October 30, 2016

1 Side and diagonal numbers

Heath [17, pp. 117–118]:

$$a_1 = 1, \qquad d_1 = 1.$$

$$a_n = a_{n-1} + d_{n-1}, \qquad d_n = 2a_{n+1} + d_{n+1}.$$

$$\begin{aligned} d_n^2 - 2a_n^2 &= 4a_{n-1}^2 + 4a_{n-1}d_{n-1} + d_{n-1}^2 - 2(a_{n-1}^2 + 2a_{n-1}d_{n-1} + d_{n-1}^2) \\ &= 4a_{n+1}^2 + 4a_{n+1}d_{n+1} + d_{n+1}^2 - 2a_{n-1}^2 - 4a_{n-1}d_{n-1} - 2d_{n-1}^2 \\ &= 2a_{n-1}^2 - d_{n-1}^2 \\ &= -(d_{n-1}^2 - 2a_{n-1}^2). \end{aligned}$$

As $d_1^2 - 2a_1^2 = -1$,

$$d_n^2 - 2a_n^2 = (-1)^n.$$

$$\begin{pmatrix} a_n \\ d_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ d_{n-1} \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ d_{n-1} \end{pmatrix}.$$

$$\begin{pmatrix} a_{n+1} \\ d_{n+1} \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$A = PDP^{-1}$.

$$D = \begin{pmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 1 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}.$$

$$A^n = \begin{pmatrix} \frac{(1-\sqrt{2})^n}{2} + \frac{(1+\sqrt{2})^n}{2} & -\frac{(1-\sqrt{2})^n}{2\sqrt{2}} + \frac{(1+\sqrt{2})^n}{2\sqrt{2}} \\ -\frac{(1-\sqrt{2})^n}{\sqrt{2}} + \frac{(1+\sqrt{2})^n}{\sqrt{2}} & \frac{(1-\sqrt{2})^n}{2} + \frac{(1+\sqrt{2})^n}{2}. \end{pmatrix}.$$

$$\begin{pmatrix} a_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(1-\sqrt{2}\right)^n - \frac{\left(1-\sqrt{2}\right)^n}{2\sqrt{2}} + \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{\left(1+\sqrt{2}\right)^n}{2\sqrt{2}} \\ \frac{1}{2}\left(1-\sqrt{2}\right)^n - \frac{\left(1-\sqrt{2}\right)^n}{\sqrt{2}} + \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{\left(1+\sqrt{2}\right)^n}{\sqrt{2}} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} a_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} a_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \qquad \begin{pmatrix} a_4 \\ d_4 \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \end{pmatrix}, \qquad \begin{pmatrix} a_5 \\ d_5 \end{pmatrix} = \begin{pmatrix} 29 \\ 41 \end{pmatrix}.$$

2 Diophantus

If $x^2 - Ay^2 = 1$ and $y = m(x+1)$ for rational m, then $y^2 = m^2(x^2+2x+1)$, then $x^2 - Am^2(x^2+2x+1) = 1$, then $(Am^2-1)x^2 + 2Am^2x + Am^2 + 1 = 0$. Write $(x+1)(px+q) = (Am^2-1)x^2 + 2Am^2x + Am^2 + 1$. Then $px^2 + (p+q)x + q = (Am^2-1)x^2 + 2Am^2x + Am^2 + 1$. Then $p = Am^2 - 1$, $q = Am^2 + 1$, and so $p + q = Am^2 - 1 + Am^2 + 1 = 2Am^2$. Thus if $x \neq -1$ then $px + q = 0$ and hence $(Am^2-1)x + Am^2 + 1 = 0$. Hence $(Am^2-1)x = -(Am^2+1)$ and so $x = -\frac{Am^2+1}{Am^2-1}$. Thus

$$y = m(x+1) = m\left(-\frac{Am^2+1}{Am^2-1}+1\right)\frac{m}{Am^2-1}(-Am^2-1+Am^2-1) = \frac{-2m}{Am^2-1}.$$

Therefore for rational m,

$$x = -\frac{Am^2+1}{Am^2-1}, \qquad y = \frac{-2m}{Am^2-1}$$

satisfy $x^2 - Ay^2 = 1$. cf. Heath [17, pp. 68–69], Nesselmann [24, p. 331].

Diophantus V.11: $30x^2 + 1 = y^2$. Say $y = 5x + 1$. $y^2 = 25x^2 + 10x + 1$. Then $5x^2 - 10x = 0$, so $x = 0$ or $5x - 10 = 0$, i.e. $x = 0$ or $x = 2$. Hence $x = 0, y = 1$ and $x = 2, y = 11$ satisfy $30x^2 + 1 = y^2$.

Diophantus V.14 [17, pp. 211–212]. $34y^2 + 1 = x^2$. Say $x = 6y - 1$. $x^2 = 36y^2 - 12y + 1$. Then $2y^2 - 12y = 0$, i.e. $y(y-6) = 0$ so $y = 0$ or $y = 6$. Then $x = -1, y = 0$ and $x = 35, y = 6$ satisfy $34y^2 + 1 = x^2$.

3 Fermat

Fermat, February 1657 [31, p. 29]:

> Given any number not a square, then there are an infinite number of squares which, when multiplied by the given number, make a square when unity is added.
>
> Example. Given 3, a nonsquare number; this number multiplied by the square number 1, and 1 being added, produces 4, which is a square.
>
> Moreover, the same 3 multiplied by the square 16, with 1 added makes 49, which is a square.
>
> And instead of 1 and 16, an infinite number of squares may be found showing the same property; I demand, however, a general rule, any number being given which is not a square.
>
> It is sought, for example, to find a square which when multiplied into 149, 109, 433, etc., becomes a square when unity is added.

4 Wallis

Wallis [1, p. 546].
Stedall [29]

5 Brouncker

Weil [32, pp. 92–99].

6 Ozanam

Ozanam [25, pp. 503–516], Liv. III, Quest. XXVI.

7 Continued fractions

Let

$$[a_0, a_1] = a_0 + \frac{1}{a_1}$$

and

$$[a_0, \ldots, a_{n-1}, a_n] = \left[a_0, \ldots, a_{n-2}, a_{n-1} + \frac{1}{a_n}\right].$$

Then for $1 \leq m < n$,

$$[a_0, \ldots, a_n] = [a_0, \ldots, a_{m-1}, [a_m, \ldots, a_n]].$$

Define

$$p_0 = a_0, \qquad q_0 = 1, \qquad p_1 = a_1 a_0 + 1, \qquad q_1 = a_1$$

and for $n \geq 2$,

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2}.$$

Hardy and Wright [13, p. 130, Theorem 149]. For $n \geq 0$,

$$[a_0, \ldots, a_n] = \frac{p_n}{q_n}.$$

For $n \geq 1$,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

For $n \geq 2$,

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

For $n \geq 0$ let

$$a'_n = [a_n, a_{n+1}, \ldots].$$

For $x = [a_0, a_1, \ldots]$,

$$x = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}}, \qquad n \geq 2.$$

Hardy and Wright [13, p. 144, Theorem 176]. A continued fraction $[a_0, a_1, \ldots]$ is said to be **periodic** if there is some $L \geq 0$ and some $k \geq 1$ such that $a_{l+k} = a_l$ for all $l \geq L$.

Theorem 1. If $x = [a_0, a_1, \ldots]$ is a periodic continued fraction, then x is a quadratic surd.

Proof. Let

$$\overline{a_L, \ldots, a_{L+k-1}} = [a_L, a_{L+1}, \ldots] = a'_L.$$

Thus

$$\begin{aligned}[a_0, \ldots, a_{L-1}, a_L, a_{L+1}, \ldots] &= [a_0, \ldots, a_{L-1}, a'_L] \\ &= [a_0, \ldots, a_{L-1}, \overline{a_L, \ldots, a_{L+k-1}}].\end{aligned}$$

As $a_{L+k} = a_L, a_{L+k+1} = a_{L+1}, \ldots$,

$$\begin{aligned}a'_L &= [a_L, a_{L+1}, \ldots] \\ &= [a_L, a_{L+1}, \ldots, a_{L+k-1}, a_L, a_{L+1}, \ldots] \\ &= [a_L, a_{L+1}, \ldots, a_{L+k-1}, a'_L].\end{aligned}$$

Let

$$\frac{p'}{q'} = [a_L, a_{L+1}, \ldots, a_{L+k-1}], \qquad \frac{p''}{q''} = [a_L, a_{L+1}, \ldots, a_{L+k-2}].$$

For $t = [a_L, a_{L+1}, \ldots, a_{L+k-1}, a'_L]$,

$$t = \frac{a'_L p' + p''}{a'_L q' + q''} = \frac{p't + p''}{q't + q''}.$$

Hence $q't^2 + q''t = p't + p''$, so $q't^2 + (q'' - p')t - p'' = 0$. For $x = [a_0, a_1, \ldots]$,

$$x = \frac{a'_L p_{L-1} + p_{L-2}}{a'_L q_{L-1} + q_{L-2}}.$$

Then

$$a'_L = \frac{p_{L-2} - q_{L-2}x}{q_{L-1}x - p_{L-1}}.$$

Thus, with $t = a'_L$,

$$q' \left(\frac{p_{L-2} - q_{L-2}x}{q_{L-1}x - p_{L-1}} \right)^2 + (q'' - p') \frac{p_{L-2} - q_{L-2}x}{q_{L-1}x - p_{L-1}} - p'' = 0.$$

Then

$$q'(p_{L-2} - q_{L-2}x)^2 + (q'' - p')(p_{L-2} - q_{L-2}x)(q_{L-1}x - p_{L-1}) - p''(q_{L-1}x - p_{L-1})^2 = 0.$$

Therefore there are integers a, b, c such that

$$ax^2 + bx + c = 0.$$

This means that x is a quadratic surd, as x is irrational. $\square$

Example. Say $x = [3, 2, 7, 4, \overline{5, 1, 12}]$. $L = 4, k = 3$.

$$\frac{p'}{q'} = [5, 1, 12] = \frac{77}{13}, \qquad \frac{p''}{q''} = [5, 1] = \frac{6}{1}.$$

$$\frac{p_{L-1}}{q_{L-1}} = \frac{p_3}{q_3} = [3, 2, 7, 4] = \frac{215}{62}, \qquad \frac{p_{L-2}}{q_{L-2}} = \frac{p_2}{q_2} = [3, 2, 7] = \frac{52}{15}.$$

Then

$$\begin{aligned}
&q'(p_{L-2} - q_{L-2}x)^2 + (q'' - p')(p_{L-2} - q_{L-2}x)(q_{L-1}x - p_{L-1}) - p''(q_{L-1}x - p_{L-1})^2 \\
=&13(52 - 15x)^2 + (1 - 77)(52 - 15x)(62x - 215) - 6(62x - 215)^2 \\
=&50541x^2 - 350444x + 607482.
\end{aligned}$$

Hence $x = [3, 2, 7, 4, \overline{5, 1, 12}]$ satisfies

$$50541x^2 - 350444x + 607482 = 0.$$

In fact,

$$x = \frac{175222 + \sqrt{1522}}{50541}.$$

Hardy and Wright [13, p. 144, Theorem 177].

Theorem 2. If x is a quadratic surd, then the continued fraction of x is periodic.

Example. Say $x^2 = 218$. $14^2 = 196$.

$$\sqrt{218} = 14 + \sqrt{218} - 14 = 14 + \cfrac{1}{\cfrac{1}{\sqrt{218} - 14}}.$$

$$(\sqrt{218} - 14)(\sqrt{218} + 14) = 218 - 196 = 22, \qquad \frac{1}{\sqrt{218} - 14} = \frac{\sqrt{218} + 14}{22}.$$

We do not need to compute the decimal expansion of $\sqrt{218}$; we merely have to calculate $\lfloor \frac{\sqrt{218}+14}{22} \rfloor$. Using $14 < \sqrt{218} < 15$,

$$\frac{1}{\sqrt{218} - 14} = 1 + \frac{\sqrt{218} + 14}{22} - 1 = 1 + \frac{\sqrt{218} - 8}{22}.$$

Then

$$\sqrt{218} = 14 + \cfrac{1}{1 + \cfrac{\sqrt{218} - 8}{22}} = 14 + \cfrac{1}{1 + \cfrac{1}{\cfrac{22}{\sqrt{218} - 8}}}$$

$$(\sqrt{218}-8)(\sqrt{218}+8) = 218-64 = 154, \qquad \frac{1}{\sqrt{218}-8} = \frac{\sqrt{218}+8}{154}.$$

Then

$$\frac{22}{\sqrt{218}-8} = \frac{22\sqrt{218}+176}{154}.$$

Using that $14 < \sqrt{218} < 15$,

$$\frac{22}{\sqrt{218}-8} = 3 + \frac{22\sqrt{218}+176}{154} - 3 = 3 + \frac{22\sqrt{218}-286}{154}.$$

Then

$$\sqrt{218} = 14 + \cfrac{1}{1+\cfrac{1}{3+\cfrac{22\sqrt{218}-286}{154}}} = 14 + \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{\cfrac{154}{22\sqrt{218}-286}}}}.$$

$$(22\sqrt{218}-286)(22\sqrt{218}+286) = 22^2\cdot 218 - 286^2 = 23716,$$

$$\frac{1}{22\sqrt{218}-286} = \frac{22\sqrt{218}+286}{23716}.$$

$$\frac{154}{22\sqrt{218}-286} = \frac{3388\sqrt{218}+44044}{23716}.$$

Using $14 < \sqrt{218} < 15$,

$$\frac{154}{22\sqrt{218}-286} = 3 + \frac{3388\sqrt{218}+44044}{23716} - 3 = 3 + \frac{3388\sqrt{218}-27104}{23716}.$$

Then

$$\sqrt{218} = 14 + \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3+\cfrac{3388\sqrt{218}-27104}{23716}}}} = 14 + \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{\cfrac{23716}{3388\sqrt{218}-27104}}}}}.$$

$$(3388\sqrt{218}-27104)(3388\sqrt{218}+27104) = 3388^2\cdot 218 - 27104^2 = 1767695776,$$

$$\frac{1}{3388\sqrt{218}-27104} = \frac{3388\sqrt{218}+27104}{1767695776}.$$

$$\frac{23716}{3388\sqrt{218}-27104} = 23716\cdot\frac{3388\sqrt{218}+27104}{1767695776}.$$

Using $14 < \sqrt{218} < 15$,

$$\begin{aligned}\frac{23716}{3388\sqrt{218}-27104} &= 1 + 23716 \cdot \frac{3388\sqrt{218}+27104}{1767695776} - 1 \\ &= 1 + \frac{80349808\sqrt{218}-1124897312}{1767695776}.\end{aligned}$$

Then

$$\begin{aligned}\sqrt{218} &= 14 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{80349808\sqrt{218}-1124897312}{1767695776}}}}} \\ &= 14 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{1767695776}{80349808\sqrt{218}-1124897312}}}}}}.\end{aligned}$$

$$(80349808\sqrt{218}-1124897312)(80349808\sqrt{218}+1124897312) = 142034016204011008.$$

$$\frac{1767695776}{80349808\sqrt{218}-1124897312} = 1767695776 \cdot \frac{80349808\sqrt{218}+1124897312}{142034016204011008}.$$

Using $14 < \sqrt{218} < 15$, the floor of the above quantity is 28. Hence

$$\begin{aligned}\frac{1767695776}{80349808\sqrt{218}-1124897312} &= 28 + \frac{142034016204011008\sqrt{218}-1988476226856154112}{142034016204011008} \\ &= 28 + \sqrt{218} - 14.\end{aligned}$$

Then

$$\sqrt{218} = 14 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{28 + \sqrt{218} - 14}}}}}$$

Thus for $x = \sqrt{218}$,

$$x - 14 = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{28 + x - 14}}}}}.$$

Thus for $t = x - 14$,

$$t = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{28 + t}}}}}.$$

Therefore $t = [0, \overline{1, 3, 3, 1, 28}]$. Hence $x = 14 + t = [14, \overline{1, 3, 3, 1, 28}]$:

$$\sqrt{218} = [14, \overline{1, 3, 3, 1, 28}].$$

8 Euler

Euler, *Algebra* [10], Part II, Chapter VII.

9 Lagrange

Konen [20, pp. 75–77].

10 Chakravala

Hankel [12, pp. 200–203]

Strachey [30, pp. 36–53]. Dickson [5, pp. 349–350].

Colebrooke [3, pp. 170–184]

Colebrooke [3, pp. 363–372]

Datta and Singh [4, II, pp. 93–99]

Datta and Singh [4, II, pp. 146–161]

Datta and Singh [4, II, pp. 161–172]

Suppose that p_n, q_n are relatively prime and

$$Aq_n^2 + s_n = p_n^2.$$

If d is a common factor of q_n and s_n then $d \mid p_n^2$, so d is a common factor of p_n^2 and q_n^2, which implies that p_n and q_n have a common factor, a contradiction.

Therefore q_n and s_n are relatively prime. Because q_n and s_n are relatively prime, by the Kuttaka algorithm there are some ρ_n, ρ'_n satisfying $-q_n\rho_n + s_n\rho'_n = p_n$. For $r_n = \rho_n + k_n s_n$, $r'_n = \rho'_n + k_n q_n$,

$$\begin{aligned} -q_n r_n + s_n r'_n &= -q_n(\rho_n + k_n s_n) + s_n(\rho'_n + k_n q_n) \\ &= -q_n\rho_n - k_n q_n s_n + s_n\rho'_n + k_n q_n s_n \\ &= -q_n\rho_n + s_n\rho'_n \\ &= p_n. \end{aligned}$$

Take $r_n < \sqrt{A} < r_n + |s_n|$. $r'_n = \frac{p_n + q_n r_n}{s_n}$. Let

$$q_{n+1} = r'_n, \qquad p_{n+1} = \frac{p_n q_{n+1} - 1}{q_n}, \qquad s_{n+1} = p_{n+1}^2 - A q_{n+1}^2.$$

Example. $69y^2 + 1 = x^2$. $A = 69$.

$Aq_0^2 + s_0 = p_0^2$: $p_0 = 8, q_0 = 1, s_0 = -5$.

$p_0 + q_0\rho_0 = \rho'_0 s_0$ is equivalent to $8 + \rho_0 = -5\rho'_0$. It is satisfied by $\rho_0 = -8, \rho'_0 = 0$. Take $r_0 = -8 - 5k_0 = 7$. $r'_0 = \frac{p_0 + q_0 r_0}{s_0} = \frac{8+1\cdot 7}{-5} = -3$.

$q_1 = -3$.

$$p_1 = \frac{p_0 q_1 - 1}{q_0} = \frac{8 \cdot -3 - 1}{1} = -25.$$

$$s_1 = p_1^2 - Aq_1^2 = 4.$$

$p_1 + q_1\rho_1 = \rho'_1 s_1$ is equivalent to $-25 - 3\rho_1 = 4\rho'_1$. This is satisfied by $\rho_1 = 1, \rho'_1 = -7$. Take $r_1 = 1 + 4k_1 = 5$. Then $r'_1 = \frac{p_1 + q_1 r_1}{s_1} = \frac{-25 - 3\cdot 5}{4} = -10$.

$q_2 = -10$.

$$p_2 = \frac{p_1 q_2 - 1}{q_1} = \frac{-25 \cdot -10 - 1}{-3} = -83.$$

$$s_2 = p_2^2 - Aq_2^2 = -11.$$

$p_2 + q_2\rho_2 = \rho'_2 s_2$ is equivalent to $-83 - 10\rho_2 = -11\rho'_2$. This is satisfied by $\rho_2 = 6, \rho'_2 = 13$. Take $r_2 = 6 - 11k_2 = 6$. Then $r'_2 = \frac{p_2 + q_2 r_2}{s_2} = \frac{-83 - 10\cdot 6}{-11} = 13$.

$q_3 = 13$.

$$p_3 = \frac{p_2 q_3 - 1}{q_2} = \frac{-83 \cdot 13 - 1}{-10} = 108.$$

$$s_3 = p_3^2 - Aq_3^2 = 3.$$

$p_3 + q_3\rho_3 = \rho'_3 s_3$ is equivalent to $108 + 13\rho_3 = 3\rho'_3$. This is satisfied by $\rho_3 = 0, \rho'_3 = 36$. Take $r_3 = 36 + 3k_3 = 6$. Then $r'_3 = \frac{p_3 + q_3 r_3}{s_3} = \frac{108 + 13\cdot 6}{3} = 62$.

$q_4 = 62$.

$$p_4 = \frac{p_3 q_4 - 1}{q_3} = \frac{108 \cdot 62 - 1}{13} = 515.$$

$$s_4 = p_4^2 - Aq_4^2 = -11.$$

$p_4 + q_4\rho_4 = \rho'_4 s_4$ is equivalent to $515 + 62\rho_4 = -11\rho'_4$. This is satisfied by $\rho_4 = 5, \rho'_4 = -75$. Take $r_4 = 5 - 11k_4 = 5$. Then $r'_4 = \frac{p_4 + q_4 r_4}{s_4} = \frac{515 + 62\cdot 5}{-11} = -75$.

$q_5 = -75$.

$$p_5 = \frac{p_4 q_5 - 1}{q_4} = \frac{515 \cdot -75 - 1}{62} = -623.$$

$$s_5 = p_5^2 - Aq_5^2 = 4.$$

$p_5 + q_5\rho_5 = \rho_5' s_5$ is equivalent to $-623 - 75\rho_5 = 4\rho_5'$. This is satisfied by $\rho_5 = 3, \rho_5' = -212$. Take $r_5 = 3 + 4k_5 = 7$. Then $r_5' = \frac{p_5+q_5r_5}{s_5} = \frac{-623-75\cdot 7}{4} = -287$.

$q_6 = -287$.

$$p_6 = \frac{p_5 q_6 - 1}{q_5} = \frac{-623 \cdot -287 - 1}{-75} = -2384.$$

$$s_6 = p_6^2 - Aq_6^2 = -5.$$

$p_6 + q_6\rho_6 = \rho_6' s_6$ is equivalent to $-2384 - 287\rho_6 = -5\rho_6'$. This is satisfied by $\rho_6 = 3, \rho_6' = 649$. Take $r_6 = 3 - 5k = 8$. Then $r_6' = \frac{p_6+q_6r_6}{s_6} = \frac{-2384-287\cdot 8}{-5} = 936$.

$q_7 = 936$.

$$p_7 = \frac{p_6 q_7 - 1}{q_6} = \frac{-2384 \cdot 936 - 1}{-287} = 7775.$$

$$s_7 = p_7^2 - Aq_7^2 = 1.$$

Therefore

$$7775^2 - 69 \cdot 936^2 = 1.$$

Thus $\sqrt{69} \sim \frac{7775}{936}$.

Example. $91y^2 + 1 = x^2$. $A = 91$.

$Aq_0^2 + s_0 = p_0^2$: $p_0 = 10$, $q_0 = 1$, $s_0 = 9$.

$p_0 + q_0\rho_0 = \rho_0' s_0$ is equivalent to $10 + \rho_0 = 9\rho_0'$. This is satisfied by $\rho_0 = -10$, $\rho_0' = 0$. Take $r_0 = -10 + 9k_0 = 8$. Then $r_0' = \frac{p_0+q_0r_0}{s_0} = \frac{10+1\cdot 8}{9} = 2$.

$q_1 = 2$.

$$p_1 = \frac{p_0 q_1 - 1}{q_0} = \frac{10 \cdot 2 - 1}{1} = 19.$$

$$s_1 = p_1^2 - Aq_1^2 = -3.$$

$p_1 + q_1\rho_1 = \rho_1' s_1$ is equivalent with $19 + 2\rho_1 = -3\rho_1'$. This is satisfied by $\rho_1 = 1, \rho_1' = -7$. Take $r_1 = 1 - 3k_1 = 7$. Then $r_1' = \frac{p_1+q_1r_1}{s_1} = \frac{19+2\cdot 7}{-3} = -11$.

$q_2 = -11$.

$$p_2 = \frac{p_1 q_2 - 1}{q_1} = \frac{19 \cdot -11 - 1}{2} = -105.$$

$$s_2 = p_2^2 - Aq_2^2 = 14.$$

$p_2 + q_2\rho_2 = \rho_2' s_2$ is equivalent with $-105 - 11\rho_2 = 14\rho_2'$. This is satisfied by $\rho_2 = 7, \rho_2' = -13$. Take $r_2 = 7, r_2' = -13$.

$q_3 = -13$.

$$p_3 = \frac{p_2 q_3 - 1}{q_2} = \frac{-105 \cdot -13 - 1}{-11} = -124.$$

$$s_3 = p_3^2 - Aq_3^2 = -3.$$

$p_3+q_3\rho_3=\rho_3's_3$ is equivalent with $-124-13\rho_3=-3\rho_3'$. This is satisfied by $\rho_3=2, \rho_3'=50$. Take $r_3=2-3k_3=8$. Then $r_3'=\frac{p_3+q_3r_3}{s_3}=\frac{-124-13\cdot 8}{-3}=76$.

$q_4=76$.

$$p_4=\frac{p_3q_4-1}{q_3}=\frac{-124\cdot 76-1}{-13}=725.$$

$$s_4=p_4^2-Aq_4^2=9.$$

$p_4+q_4\rho_4=\rho_4's_4$ is equivalent with $725+76\rho_4=9\rho_4'$. This is satisfied by $\rho_4=1, \rho_4'=89$. Take $r_4=1$, $r_4'=89$.

$q_5=89$.

$$p_5=\frac{p_4q_5-1}{q_4}=\frac{725\cdot 89-1}{76}=849.$$

$$s_5=p_5^2-Aq_5^2=-10.$$

$p_5+q_5\rho_5=\rho_5's_5$ is equivalent with $849+89\rho_5=-10\rho_5'$. This is satisfied by $\rho_5=9, \rho_5'=-165$. Take $r_5=9, r_5'=-165$.

$q_6=-165$.

$$p_6=\frac{p_5q_6-1}{q_5}=\frac{849\cdot -165-1}{89}=-1574.$$

$$s_6=p_6^2-Aq_6^2=1.$$

Therefore

$$1574^2-91\cdot 165^2=1.$$

Thus $\sqrt{91}\sim\frac{1574}{165}$.

Example. $109y^2+1=x^2$. $A=109$.

$Ay_0^2+s_0=x_0^2$: $x_0=10$, $y_0=1$, $s_0=-9$.

$x_0+y_0\rho_0=s_0\rho_0'$ is equivalent to $10+\rho_0=-9\rho_0'$. This is satisfied by $\rho_0=-10$, $\rho_0'=0$. Take $r_0=-10+9k_0=8$. Then $r_0'=\frac{x_0+y_0r_0}{s_0}=\frac{10+1\cdot 8}{-9}=-2$.

$y_1=-2$.

$$x_1=\frac{x_0y_1-1}{y_0}=\frac{10\cdot -2-1}{1}=-21.$$

$$s_1=x_1^2-Ay_1^2=5.$$

$x_1+y_1\rho_1=\rho_1's_1$ is equivalent with $-21-2\rho_1=5\rho_1'$. This is satisfied by $\rho_1=2, \rho_1'=-5$. Take $r_1=2+5k_1=7$. Then $r_1'=\frac{x_1+y_1r_1}{s_1}=\frac{-21-2\cdot 7}{5}=-7$.

$y_2=-7$.

$$x_2=\frac{x_1y_2-1}{y_1}=\frac{-21\cdot -7-1}{-2}=-73.$$

$$s_2=x_2^2-Ay_2^2=-12.$$

$x_2+y_2\rho_2=\rho_2's_2$ is equivalent with $-73-7\rho_2=-12\rho_1'$. This is satisfied by $\rho_2=5, \rho_2'=9$. Take $r_2=5, r_2'=9$.

$y_3=9$.

$$x_3=\frac{x_2y_3-1}{y_2}=\frac{-73\cdot 9-1}{-7}=94.$$

$$s_3 = x_3^2 - Ay_3^2 = 7.$$

$x_3 + y_3\rho_3 = \rho_3' s_3$ is equivalent with $94 + 9\rho_3 = 7\rho_3'$. This is satisfied by $\rho_3 = 2, \rho_3' = 16$. Take $r_3 = 2 + 7k_3 = 9$. Then $r_3' = \frac{x_3+y_3r_3}{s_3} = \frac{94+9\cdot 9}{7} = 25$.

$y_4 = 25$.

$$x_4 = \frac{x_3y_4 - 1}{y_3} = \frac{94 \cdot 25 - 1}{9} = 261.$$

$$s_4 = x_4^2 - Ay_4^2 = -4.$$

$x_4 + y_4\rho_4 = \rho_4' s_4$ is equivalent with $261 + 25\rho_4 = -4\rho_4'$. This is satisfied by $\rho_4 = 3, \rho_4' = -84$. Take $r_4 = 3 - 4k_4 = 7$. Then $r_4' = \frac{x_4+y_4r_4}{s_4} = \frac{261+25\cdot 7}{-4} = -109$.

$y_5 = -109$.

$$x_5 = \frac{x_4y_5 - 1}{y_4} = \frac{261 \cdot -109 - 1}{25} = -1138.$$

$$s_5 = x_5^2 - Ay_5^2 = 15.$$

$x_5 + y_5\rho_5 = \rho_5' s_5$ is equivalent with $-1138 - 109\rho_5 = 15\rho_5'$. This is satisfied by $\rho_5 = 8, \rho_5' = -134$. Take $r_5 = 8, r_5' = -134$.

$y_6 = -134$.

$$x_6 = \frac{x_5y_6 - 1}{y_5} = \frac{-1138 \cdot -134 - 1}{-109} = -1399.$$

$$s_6 = x_6^2 - Ay_6^2 = -3.$$

$x_6 + y_6\rho_6 = \rho_6' s_6$ is equivalent with $-1399 - 134\rho_6 = -3\rho_6'$. This is satisfied by $\rho_6 = 1, \rho_6' = 511$. Take $r_6 = 1 - 3k_6 = 10$. Then $r_6' = \frac{x_6+y_6r_6}{s_6} = \frac{-1399-134\cdot 10}{-3} = 913$.

$y_7 = 913$.

$$x_7 = \frac{x_6y_7 - 1}{y_6} = \frac{-1399 \cdot 913 - 1}{-134} = 9532.$$

$$s_7 = x_7^2 - Ay_7^2 = 3.$$

$x_7 + y_7\rho_7 = \rho_7' s_7$ is equivalent with $9532 + 913\rho_7 = 3\rho_7'$. This is satisfied by $\rho_7 = 2, \rho_7' = 3786$. Take $r_7 = 2 + 3k_7 = 8$. Then $r_7' = \frac{x_7+y_7r_7}{s_7} = \frac{9532+913\cdot 8}{3} = 5612$.

$y_8 = 5612$.

$$x_8 = \frac{x_7y_8 - 1}{y_7} = \frac{9532 \cdot 5612 - 1}{913} = 58591.$$

$$s_8 = x_8^2 - Ay_8^2 = -15.$$

$x_8 + y_8\rho_8 = \rho_8' s_8$ is equivalent with $58591 + 5612\rho_8 = -15\rho_8'$. This is satisfied by $\rho_8 = 7, \rho_7' = -6525$. Take $r_8 = 7, r_8' = -6525$.

$y_9 = -6525$.

$$x_9 = \frac{x_8 y_9 - 1}{y_8} = \frac{58591 \cdot -6525 - 1}{5612} = -68123.$$

$$s_9 = x_9^2 - Ay_9^2 = 4.$$

$x_9 + y_9\rho_9 = \rho_9' s_9$ is equivalent with $-68123 - 6525\rho_9 = 4\rho_9'$. This is satisfied by $\rho_9 = 1, \rho_9' = -18662$. Take $r_9 = 1 + 4k_9 = 9$. Then $r_9' = \frac{x_9 + y_9 r_9}{s_9} = \frac{-68123 - 6525 \cdot 9}{4} = -31712$.

$y_{10} = -31712$.

$$x_{10} = \frac{x_9 y_{10} - 1}{y_9} = \frac{-68123 \cdot (-31712) - 1}{-6525} = -331083.$$

$$s_{10} = x_{10}^2 - Ay_{10}^2 = -7.$$

$x_{10} + y_{10}\rho_{10} = \rho_{10}' s_{10}$ is equivalent with $-331083 - 31712\rho_{10} = -7\rho_{10}'$. This is satisfied by $\rho_{10} = 5, \rho_{10}' = 69949$. Take $r_{10} = 5, r_{10}' = 69949$.

$y_{11} = 69949$.

$$x_{11} = \frac{x_{10} y_{11} - 1}{y_{10}} = \frac{-331083 \cdot 69949 - 1}{-31712} = 730289.$$

$$s_{11} = x_{11}^2 - Ay_{11}^2 = 12.$$

$x_{11} + y_{11}\rho_{11} = \rho_{11}' s_{11}$ is equivalent with $730289 + 69949\rho_{11} = 12\rho_{11}'$. This is satisfied by $\rho_{11} = 7, \rho_{11}' = 101661$. Take $r_{11} = 5, r_{10}' = 101661$.

$y_{12} = 101661$.

$$x_{12} = \frac{x_{11} y_{12} - 1}{y_{11}} = \frac{730289 \cdot 101661 - 1}{69949} = 1061372.$$

$$s_{12} = x_{12}^2 - Ay_{12}^2 = -5.$$

$x_{12} + y_{12}\rho_{12} = \rho_{12}' s_{12}$ is equivalent with $1061372 + 101661\rho_{12} = -5\rho_{12}'$. This is satisfied by $\rho_{12} = 3, \rho_{12}' = -273271$. Take $r_{12} = 3 - 5k_{12} = 8$. Then $r_{12}' = \frac{x_{12} + y_{12} r_{12}}{s_{12}} = \frac{1061372 + 101661 \cdot 8}{-5} = -374932$.

$y_{13} = -374932$.

$$x_{13} = \frac{x_{12} y_{13} - 1}{y_{12}} = \frac{1061372 \cdot (-374932) - 1}{101661} = -3914405.$$

$$s_{13} = x_{13}^2 - Ay_{13}^2 = 9.$$

$x_{13} + y_{13}\rho_{13} = \rho_{13}' s_{13}$ is equivalent with $-3914405 - 374932\rho_{13} = 9\rho_{13}'$. This is satisfied by $\rho_{13} = 1, \rho_{13}' = -476593$. Take $r_{13} = 1 + 9k_{13} = 10$. Then $r_{13}' = \frac{x_{13} + y_{13} r_{13}}{s_{13}} = \frac{-3914405 - 374932 \cdot 10}{9} = -851525$.

$y_{14} = -851525$.

$$x_{14} = \frac{x_{13} y_{14} - 1}{y_{13}} = \frac{-3914405 \cdot (-851525) - 1}{-374932} = -8890182.$$

$$s_{14} = x_{14}^2 - Ay_{14}^2 = -1.$$

$x_{14} + y_{14}\rho_{14} = \rho'_{14}s_{14}$ is equivalent with $-8890182 - 851525\rho_{14} = -\rho'_{14}$. This is satisfied by $\rho_{14} = 0, \rho'_{14} = 8890182$. Take $r_{14} = -k_{14} = 10$. Then $r'_{14} = \frac{x_{14}+y_{14}r_{14}}{s_{14}} = \frac{-8890182-851525\cdot 10}{-1} = 17405432$.

$y_{15} = 17405432$.

$$x_{15} = \frac{x_{14}y_{15} - 1}{y_{14}} = \frac{-8890182 \cdot 17405432 - 1}{-851525} = 181718045.$$

$$s_{15} = x_{15}^2 - Ay_{15}^2 = 9.$$

$r_{15} = 8, r'_{15} = 35662389$.

$y_{16} = 35662389$.

$$x_{16} = \frac{x_{15}y_{16} - 1}{y_{15}} = \frac{35662389 \cdot 35662389 - 1}{17405432} = 372326272.$$

$$s_{16} = x_{16}^2 - Ay_{16}^2 = -5.$$

$\rho_{16} = 2, \rho'_2 = -88730210$. $r_{16} = 2 - 5k_{16} = 7$. Then $r'_{16} = -124392599$.

$y_{17} = -124392599$.

$$x_{17} = -1298696861.$$

$$s_{17} = 12.$$

$r_{18} = 5, r'_{18} = -160054988$.

$y_{18} = -160054988$.

$$x_{18} = -1671023133.$$

$$s_{18} = -7.$$

$\rho_{19} = 2, \rho'_{19} = 284447587$. Take $r_{19} = 2 - 7k_{19} = 9$. Then $r'_{19} = 444502575$.

$y_{19} = 444502575$.

$$x_{19} = 4640743127.$$

$$s_{19} = 4.$$

$\rho_{20} = 3, \rho'_{20} = 1493562713$. Take $r_{20} = 3 + 4k_{20} = 7$. Then $r'_{20} = 1938065288$.

$y_{20} = 1938065288$.

$$x_{20} = 20233995641.$$

$$s_{20} = -15.$$

$r_{21} = 8, r'_{21} = -2382567863$.

$y_{21} = -2382567863$.

$$x_{21} = -24874738768.$$

$$s_{21} = 3.$$

$\rho_{22} = 1, \rho'_{22} = -9085768877$. Take $r_{22} = 1 + 3k_{22} = 10$. Then $r'_{22} = -16233472466$.

$y_{22} = -16233472466.$

$$x_{22} = . - 169482428249.$$

$$s_{22} = -3.$$

$\rho_{23} = 2, \rho'_{23} = 67316457727.$ Take $r_{23} = 2 - 3k_{23} = 8.$ Then $r'_{23} = 99783402659.$

$y_{23} = 99783402659.$

$$x_{23} = 1041769308262.$$

$$s_{23} = 15.$$

$r_{24} = 7, r'_{24} = 116016875125.$

$y_{24} = 116016875125.$

$$x_{24} = 1211251736511.$$

$$s_{24} = -4.$$

$\rho_{25} = 1, \rho'_{25} = -331817152909.$ Take $r_{25} = 1 - 4k_{25} = 9.$ Then $r'_{25} = -563850903159.$

$y_{25} = -563850903159.$

$$x_{25} = . - 5886776254306.$$

$$s_{25} = 7.$$

$r_{26} = 5, r'_{26} = -1243718681443.$

$y_{26} = -1243718681443.$

$$x_{26} = -12984804245123.$$

$$s_{26} = . - 12.$$

$r_{27} = 7, r'_{27} = 1807569584602.$

$y_{27} = 1807569584602.$

$$x_{27} = 18871580499429.$$

$$s_{27} = 5.$$

$\rho_{28} = 3, \rho'_{28} = 4858857850647.$ Take $r_{28} = 3 + 5k_{28} = 8.$ Then $r'_{28} = 6666427435249.$

$y_{28} = 6666427435249.$

$$x_{28} = 69599545743410.$$

$$s_{28} = -9.$$

$\rho_{29} = 1, \rho'_{29} = -8473997019851.$ Take $r_{29} = 1 + 9k_{29} = 10.$ Then $r'_{29} = -15140424455100.$

$y_{29} = -15140424455100.$

$$x_{29} = -158070671986249.$$

$$s_{29} = 1.$$

Therefore

$$158070671986249^2 - 109 \cdot 15140424455100^2 = 1.$$

Thus $\sqrt{109} \sim \frac{158070671986249}{15140424455100}$.

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