Induction

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Let $\mathbb{N} = \{0, 1, 2, \ldots\}$

Theorem 1. *For all $n \in \mathbb{N}$, $6^n - 1$ is a multiple of 5.*

Proof. **Base case** For $n = 0$, $6^n - 1 = 6^0 - 1 = 0$. 0 is a multiple of 5: $0 = 0 \cdot 5$. The claim is true for $n = 0$.

Inductive step Assume the claim is true for some $n \geq 0$. That is, suppose that $6^n - 1$ is a multiple of 5. Being a multiple of 5 means that there is some $a \in \mathbb{N}$ such that $6^n - 1 = 5a$.

$$\begin{aligned} 6^{n+1} - 1 &= 6(6^n) - 1 \\ &= 6(6^n - 1 + 1) - 1 \\ &= 6(6^n - 1) + 6 - 1 \\ &= 6(6^n - 1) + 5 \\ &= 6(5a) + 5 \\ &= (6a)5 + 5 \\ &= (6a + 1)5 \end{aligned}$$

$6a + 1 \in \mathbb{N}$, so the above shows that $6^{n+1} - 1$ is a multiple of 5, completing the inductive step.

Conclusion By induction, for all $n \in \mathbb{N}$ it is true that $6^n - 1$ is a multiple of 5. □

Theorem 2. *For $n \geq 5$, $2n + 1 < 2^n$.*

Proof. **Base case** For $n = 5$, $2n+1 = 11$ and $2^n = 32$, and it is true that $11 < 32$. The claim is true when $n = 5$.

Inductive step Assume the claim is true for some $n \geq 5$. That is, suppose $2n+1 < 2^n$.

$$2(n + 1) + 1 = (2n + 1) + 2 < 2^n + 2$$

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$2^n + 2 < 2^n + 2^n$ for $n \geq 1$, and here $n \geq 5$ so this is true.

Then we have

$2(n + 1) + 1 < 2^n + 2^n$, and $2^n + 2^n = 2(2^n) = 2^{n+1}$, so

$$2(n + 1) + 1 < 2^{n+1}$$

This shows that the claim is true for $n + 1$, completing the inductive step.

Conclusion By induction, for all $n \geq 5$ it is true that $2n + 1 < 2^n$. □

Theorem 3. *For $n \geq 5$, $n^2 < 2^n$.*

Proof. **Base case** Let $n = 5$. $n^2 = 25$ and $2^n = 32$, and it is true that $25 < 32$. The claim is true when $n = 5$.

Inductive step Assume the claim is true for some $n \geq 5$. That is, suppose that $n^2 < 2^n$

$$\begin{aligned} (n+1)^2 &= n^2 + 2n + 1 && \text{expanding} \\ &< 2^n + 2n + 1 && \text{because } n^2 < 2^n && < 2^n + 2n + 1 \end{aligned}$$

We proved in the previous theorem that for $n \geq 5$, $2n + 1 < 2^n$. Therefore ,

$$2^n + 2n + 1 = 2^n + (2n + 1) < 2^n + 2^n = 2(2^n) = 2^{n+1}$$

This show that the claim is true for $n + 1$, completing the inductive step.

Conclusion By induction, for $n \geq 5$ it is true that $n^2 < 2^n$. □