## CHAP. VII.

### *Of a particular* Method, *by which the* Formula $an^2 + 1$ *becomes a* Square *in* Integers.

96. That which has been taught in the last chapter, cannot be completely performed, unless we are able to assign for any number $a$, a number $n$, such, that $an^2 + 1$ may become a square; or that we may have $m^2 = an^2 + 1$.

This equation would be easy to resolve, if we were satisfied with fractional numbers, since we should have only to make $m = 1 + \frac{np}{q}$; for, by this supposition, we have $m^2 = 1 + \frac{2np}{q} + \frac{n^2p^2}{q^2} = an^2 + 1$; in which equation, we may expunge 1 from both sides, and divide the other terms by $n$: then multiplying by $q^2$, we obtain $2pq + np^2 = anq^2$; and this equation, giving $n = \frac{2pq}{aq^2 - p^2}$, would furnish an infinite number of values for $n$: but as $n$ must be an integer number, this method will be of no use, and therefore very different means must be employed in order to accomplish our object.

97. We must begin with observing, that if we wished to have $an^2 + 1$ a square, in integer numbers, (whatever be the value of $a$), the thing required would not be possible.

For, in the first place, it is necessary to exclude all the cases, in which $a$ would be negative; next, we must exclude those also, in which $a$ would be itself a square; because then $an^2$ would be a square, and no square can become a square, in integer numbers, by being increased by unity. We are obliged, therefore, to restrict our formula to the condition, that $a$ be neither negative, nor a square; but whenever $a$ is a positive number, without being a square, it is possible to assign such an integer value of $n$, that $an^2 + 1$ may become a square: and when one such value has been found, it will be easy to deduce from it an infinite number of others, as was taught in the last chapter: but for our purpose it is sufficient to know a single one, even the least;

and this, Pell, an English writer, has taught us to find by an ingenious method, which we shall here explain.

98. This method is not such as may be employed generally, for any number $a$ whatever; it is applicable only to each particular case.

We shall therefore begin with the easiest cases, and shall first seek such a value of $n$, that $2n^2 + 1$ may be a square, or that $\sqrt{(2n^2 + 1)}$ may become rational.

We immediately see that this square root becomes greater than $n$, and less than $2n$. If, therefore, we express this root by $n + p$, it is obvious that $p$ must be less than $n$; and we shall have $\sqrt{(2n^2 + 1)} = n + p$; then, by squaring, $2n^2 + 1 = n^2 + 2np + p^2$; therefore

$$n^2 = 2np + p^2 - 1, \text{ and } n = p + \sqrt{(2p^2 - 1)}.$$

The whole is reduced, therefore, to the condition of $2p^2 - 1$ being a square; now, this is the case if $p = 1$, which gives $n = 2$, and $\sqrt{(2n + 1)} = 3$.

If this case had not been immediately obvious, we should have gone farther; and since $\sqrt{(2p^2 - 1)} > p$*, and, consequently, $n > 2p$, we should have made $n = 2p + q$; and should thus have had

$$2p + q = p + \sqrt{(2p^2 - 1)}, \text{ or } p + q = \sqrt{(2p^2 - 1)},$$

and, squaring, $p^2 + 2pq + q^2 = 2p^2 - 1$, whence

$$p^2 = 2pq + q^2 + 1,$$

which would have given $p = q + \sqrt{(2q^2 + 1)}$; so that it would have been necessary to have $2q^2 + 1$ a square; and as this is the case, if we make $q = 0$, we shall have $p = 1$, and $n = 2$, as before. This example is sufficient to give an idea of the method; but it will be rendered more clear and distinct from what follows.

99. Let $a = 3$, that is to say, let it be required to transform the formula $3n^2 + 1$ into a square. Here we shall make $\sqrt{(3n^2 + 1)} = n + p$, which gives

$$3n^2 + 1 = n^2 + 2np + p^2, \text{ and } 2n^2 = 2np + p^2 - 1;$$

whence we obtain $n = \frac{p + \sqrt{(3p^2 - 2)}}{2}$. Now, since $\sqrt{(3p^2 - 2)}$ exceeds $p$, and, consequently, $n$ is greater

* This sign, $>$, placed between two quantities, signifies that the former is greater than the latter; and when the angular point is turned the contrary way, as $<$, it signifies that the former is less than the latter.

than $\frac{2p}{2}$, or than $p$, let us suppose $n = p + q$, and we shall have

$$2p + 2q = p + \surd(3p^2 - 2), \text{ or}$$
$$p + 2q = \surd(3p^2 - 2);$$

then, by squaring $p^2 + 4pq + 4q^2 = 3p^2 - 2$; so that

$$2p^2 = 4pq + 4q^2 + 2, \text{ or } p^2 = 2pq + 2q^2 + 1, \text{ and}$$
$$p = q + \surd(3q^2 + 1).$$

Now, this formula being similar to the one proposed, we may make $q = 0$, and shall thus obtain $p = 1$, and $n = 1$; whence $\surd(3n^2 + 1) = 2$.

100. Let $a = 5$, that we may have to make a square of the formula $5n^2 + 1$, the root of which is greater than $2n$. We shall therefore suppose

$$\surd(5n^2 + 1) = 2n + p, \text{ or } 5n^2 + 1 = 4n^2 + 4np + p^2;$$

whence we obtain

$$n^2 = 4np + p^2 - 1, \text{ and } n = 2p + \surd(5p^2 - 1).$$

Now, $\surd(5p^2 - 1) > 2p$; whence it follows that $n > 4p$; for which reason, we shall make $n = 4p + q$, which gives $2p + q = \surd(5p^2 - 1)$, or $4p^2 + 4pq + q^2 = 5p^2 - 1$, and $p^2 = 4pq + q^2 + 1$; so that $p = 2q + \surd(5q^2 + 1)$; and as $q = 0$ satisfies the terms of this equation, we shall have $p = 1$, and $n = 4$; therefore $\surd(5n^2 + 1) = 9$.

101. Let us now suppose $a = 6$, that we may have to consider the formula $6n^2 + 1$, whose root is likewise contained between $2n$ and $3n$. We shall, therefore, make $\surd(6n^2 + 1) = 2n + p$, and shall have

$$6n^2 + 1 = 4n^2 + 4np + p^2, \text{ or } 2n^2 = 4np + p^2 - 1;$$

and, thence, $n = p + \frac{\surd(6p^2 - 2)}{2}$, or $n = \frac{2p + \surd(6p^2 - 2)}{2}$;

so that $n > 2p$.

If, therefore, we make $n = 2p + q$, we shall have

$$4p + 2q = 2p + \surd(6p^2 - 2), \text{ or}$$
$$2p + 2q = \surd(6p^2 - 2);$$

the squares of which are $4p^2 + 8pq + 4q^2 = 6p^2 - 2$; so that $2p^2 = 8pq + 4q^2 + 2$, and $p^2 = 4pq + 2q^2 + 1$. Lastly, $p = 2q + \surd(6q^2 + 1)$. Now, this formula resembling the first, we have $q = 0$; wherefore $p = 1$, $n = 2$, and $\surd(6n^2 + 1) = 5$.

102. Let us proceed farther, and take $a = 7$, and $7n^2 + 1 = m^2$; here we see that $m > 2n$; let us therefore make $m = 2n + p$, and we shall have

$7n^2 + 1 = 4n^2 + 4np + p^2$, or $3n^2 = 4np + p^2 - 1$; which gives $n = \frac{2p + \surd(7p^2 - 3)}{3}$. At present, since $n > \frac{4}{3}p$, and, consequently, greater than $p$, let us make $n = p + q$, and we shall have $p + 3q = \surd(7p^2 - 3)$; then, squaring both sides, $p^2 + 6pq + 9q^2 = 7p^2 - 3$, so that $6p^2 = 6pq + 9q^2 + 3$, or $2p^2 = 2pq + 3q^2 + 1$; whence we get $p = \frac{q + \surd(7q^2 + 2)}{2}$. Now, we have here $p > \frac{3q}{2}$; and, consequently, $p > q$; so that making $p = q + r$, we shall have $q + 2r = \surd(7q^2 + 2)$; the squares of which are $q^2 + 4qr + 4r^2 = 7q^2 + 2$; then $6q^2 = 4qr + 4r^2 - 2$, or $3q^2 = 2qr + 2r^2 - 1$; and, lastly, $q = \frac{r + \surd(7r^2 - 3)}{3}$.

Since now $q > r$, let us suppose $q = r + s$, and we shall have

$$2r + 3s = \surd(7r^2 - 3); \text{ then}$$
$$4r^2 + 12rs + 9s^2 = 7r^2 - 3, \text{ or}$$
$$3r^2 = 12rs + 9s^2 + 3, \text{ or}$$
$$r^2 = 4rs + 3s^2 + 1, \text{ and}$$
$$r = 2s + \surd(7s^2 + 1).$$

Now, this formula is like the first; so that making $s = 0$, we shall obtain $r = 1$, $q = 1$, $p = 2$, and $n = 3$, or $m = 8$.

But this calculation may be considerably abridged in the following manner, which may be adopted also in other cases.

Since $7n^2 + 1 = m^2$, it follows that $m < 3n$.

If, therefore, we suppose $m = 3n - p$, we shall have $7n^2 + 1 = 9n^2 - 6np + p^2$, or $2n^2 = 6np - p^2 + 1$; whence we obtain $n = \frac{3p + \surd(7p^2 + 2)}{2}$; so that $n < 3p$; for this reason we shall write $n = 3p - 2q$; and, squaring, we shall have $9p^2 - 12pq + 4q^2 = 7p^2 + 2$; or

$$2p^2 = 12pq - 4q^2 + 2, \text{ and } p^2 = 6pq - 2q^2 + 1,$$

whence results $p = 3q + \surd(7q^2 + 1)$. Here, we can at once make $q = 0$, which gives $p = 1$, $n = 3$, and $m = 8$, as before.

103. Let $a = 8$, so that $8n^2 + 1 = m^2$, and $m < 3n$. Here, we must make $m = 3n - p$, and shall have

$$8n^2 + 1 = 9n^2 - 6np + p^2, \text{ or } n^2 = 6np - p^2 + 1;$$

whence $n = 3p + \surd(8p^2 + 1)$, and this formula being already similar to the one proposed, we may make $p = 0$, which gives $n = 1$, and $m = 3$.

104. We may proceed, in the same manner, for every other number, $a$, provided it be positive and not a square, and we shall always be led, at last, to a radical quantity, such as $\sqrt{(at^2 + 1)}$, similar to the first, or given formula, and then we have only to suppose $t = 0$; for the irrationality will disappear, and by tracing back the steps, we shall necessarily find such a value of $n$, as will make $an^2 + 1$ a square.

Sometimes we quickly obtain our end; but, frequently also, we are obliged to go through a great number of operations. This depends on the nature of the number $a$; but we have no principles, by which we can foresee the number of operations that it will be necessary to perform. The process is not very long for numbers below 13, but when $a = 13$, the calculation becomes much more prolix; and, for this reason, it will be proper here to resolve that case.

105. Let therefore $a = 13$, and let it be required to find $13n^2 + 1 = m^2$. Here, as $m^2 > 9n^2$, and, consequently, $m > 3n$, let us suppose $m = 3n + p$; we shall then have $13n^2 + 1 = 9n^2 + 6np + p^2$, or $4n^2 = 6np + p^2 - 1$, and $n = \frac{3p + \sqrt{(13p^2 - 4)}}{4}$, which shews that $n > \frac{6}{4}p$, and therefore much greater than $p$. If, therefore, we make $n = p + q$, we shall have $p + 4q = \sqrt{(13p^2 - 4)}$; and, taking the squares,

$$13p^2 - 4 = p^2 + 8pq + 16q^2;$$

so that $12p^2 = 8pq + 16q^2 + 4$, or $3p^2 = 2pq + 4q^2 + 1$, and $p = \frac{q + \sqrt{(13q^2 + 3)}}{3}$. Here, $p > \frac{q + 3q}{3}$, or $p > q$; we shall proceed, therefore, by making $p = q + r$, and shall thus obtain $2q + 3r = \sqrt{(13q^2 + 3)}$; then

$$13q^2 + 3 = 4q^2 + 12qr + 9r^2, \text{ or}$$
$$9q^2 = 12qr + 9r^2 - 3, \text{ or}$$
$$3q^2 = 4qr + 3r^2 - 1;$$

which gives $q = \frac{2r + \sqrt{(13r^2 - 3)}}{3}$.

Again, since $q > \frac{2r + 3r}{3}$, or $q > r$, we shall make $q = r + s$, and we shall thus have $r + 3s = \sqrt{(13r^2 - 3)}$; or $13r^2 - 3 = r^2 + 6rs + 9s^2$, or $12r^2 = 6rs + 9s^2 + 3$, or $4r^2 = 2rs + 3s^2 + 1$; whence we obtain

$r = \frac{s + \sqrt{(13s^2 + 4)}}{4}$. But here $r > \frac{s+3s}{4}$, or $r > s$; wherefore let $r = s + t$, and we shall have $3s + 4t = \sqrt{(13s^2 + 4)}$,

$$\text{and } 13s^2 + 4 = 9s^2 + 24st + 16t^2;$$

so that $4s^2 = 24st + 16t^2 - 4$, and $s^2 = 6ts + 4t^2 - 1$; therefore $s = 3t + \sqrt{(13t^2 - 1)}$. Here we have

$$s > 3t + 3t, \text{ or } s > 6t;$$

we must therefore make $s = 6t + u$; whence $3t + u = \sqrt{(13t^2 - 1)}$, and $13t^2 - 1 = 9t^2 + 6tu + u^2$; then $4t^2 = 6tu + u^2 + 1$; and, lastly,

$$t = \frac{3u + \sqrt{(13u^2 + 4)}}{4}, \text{ or } t > \frac{6u}{4}, \text{ and } > u.$$

If, therefore, we make $t = u + v$, we shall have $u + 4v = \sqrt{(13u^2 + 4)}$, and $13u^2 + 4 = u^2 + 8uv + 16v^2$; therefore $12u^2 = 8uv + 16v^2 - 4$, or $3u^2 = 2uv + 4v^2 - 1$;

lastly, $u = \frac{v + \sqrt{(13v^2 - 3)}}{3}$, or $u > \frac{4v}{3}$, or $u > v$.

Let us, therefore, make $u = v + x$, and we shall have

$$2v + 3x = \sqrt{(13v^2 - 3)}, \text{ and}$$
$$13v^2 - 3 = 4v^2 + 12vx + 9x^2; \text{ or}$$

$9v^2 = 12vx + 9x^2 + 3$, or $3v^2 = 4vx + 3x^2 + 1$, and

$v = \frac{2x + \sqrt{(13x^2 + 3)}}{3}$; so that $v > \frac{5}{3}x$, and $> x$.

Let us now suppose $v = x + y$, and we shall have

$$x + 3y = \sqrt{(13x^2 + 3)}, \text{ and}$$
$$13x^2 + 3 = x^2 + 6xy + 9y^2, \text{ or}$$
$$12x^2 = 6xy + 9y^2 - 3, \text{ and}$$
$$4x^2 = 2xy + 3y^2 - 1; \text{ whence}$$
$$x = \frac{y + \sqrt{(13y^2 - 4)}}{4},$$

and, consequently, $x > y$. We shall, therefore, make $x = y + z$, which gives

$$3y + 4z = \sqrt{(13y^2 - 4)}, \text{ and}$$
$$13y^2 - 4 = 9y^2 + 24zy + 16z^2, \text{ or}$$
$$4y^2 = 24zy + 16z^2 + 4; \text{ therefore}$$
$$y^2 = 6yz + 4z^2 + 1, \text{ and}$$
$$y = 3z + \sqrt{(13z^2 + 1)}.$$

This formula being at length similar to the first, we may take $z = 0$, and go back as follows:

| | | |
|---|---|---|
| $z = 0,$ | $u = v + x = 3,$ | $q = r + s = 71,$ |
| $y = 1,$ | $t = u + v = 5,$ | $p = q + r = 109,$ |
| $x = y + z = 1,$ | $s = 6t + u = 33,$ | $n = p + q = 180,$ |
| $v = x + y = 2,$ | $r = s + t = 38,$ | $m = 3n + p = 649.$ |

So that 180 is the least number, after 0, which we can substitute for $n$, in order that $13n^2 + 1$ may become a square.

106. This example sufficiently shews how prolix these calculations may be in particular cases; and when the numbers in question are greater, we are often obliged to go through ten times as many operations as we had to perform for the number 13.

As we cannot foresee the numbers that will require such tedious calculations, we may with propriety avail ourselves of the trouble which others have taken; and, for this purpose, a Table is subjoined to the present chapter, in which the values of $m$ and $n$ are calculated for all numbers, $a$, between 2 and 100; so that in the cases which present themselves, we may take from it the values of $m$ and $n$, which answer to the given number $a$.

107. It is proper, however, to remark, that, for certain numbers, the letters $m$ and $n$ may be determined generally; this is the case when $a$ is greater, or less than a square, by 1 or 2; it will be proper, therefore, to enter into a particular analysis of these cases.

108. In order to this, let $a = e^2 - 2$; and since we must have $(e^2 - 2)n^2 + 1 = m^2$, it is clear that $m \angle en$; therefore we shall make $m = en - p$, from which we have

$$(e^2 - 2)n^2 + 1 = e^2n^2 - 2enp + p^2, \text{ or}$$
$$2n^2 = 2enp - p^2 + 1; \text{ therefore}$$

$n = \dfrac{ep + \sqrt{(e^2p^2 - 2p^2 + 2)}}{2}$; and it is evident that if we make $p = 1$, this quantity becomes rational, and we have $n = e$, and $m = e^2 - 1$.

For example, let $a = 23$, so that $e = 5$; we shall then have $23n^2 + 1 = m^2$, if $n = 5$, and $m = 24$. The reason of which is evident from another consideration; for if, in the case of $a = e^2 - 2$, we make $n = e$, we shall have $an^2 + 1 = e^4 - 2e^2 + 1$; which is the square of $e^2 - 1$.

109. Let $a = e^2 - 1$, or less than a square by unity. First, we must have $(e^2 - 1)n^2 + 1 = m^2$; then, because, as before, $m \angle en$, we shall make $m = en - p$; and this being done, we have

$$(e^2 - 1)n^2 + 1 = e^2n^2 - 2enp + p^2, \text{ or } n^2 = 2enp - p^2 + 1;$$

wherefore $n = ep + \sqrt{(e^2p^2 - p^2 + 1)}$. Now, the irrationality disappeared by supposing $p = 1$; so that $n = 2e$, and $m = 2e^2 - 1$. This also is evident; for, since $a = e^2 - 1$, and $n = 2e$, we find

$$an^2 + 1 = 4e^4 - 4e^2 + 1,$$

or equal to the square of $2e^2 - 1$. For example, let $a = 24$, or $e = 5$, we shall have $n = 10$, and

$$24n^2 + 1 = 2401 = (49)^2 *.$$

110. Let us now suppose $a = e^2 + 1$, or $a$ greater than a square by unity. Here we must have

$$(e^2 + 1)n^2 + 1 = m^2,$$

and $m$ will evidently be greater than $en$. Let us, therefore, write $m = en + p$, and we shall have

$$(e^2 + 1)n^2 + 1 = e^2n^2 + 2enp + p^2, \text{ or } n^2 = 2enp + p^2 - 1;$$

whence $n = ep + \sqrt{(e^2p^2 + p^2 - 1)}$. Now, we may make $p = 1$, and shall then have $n = 2e$; therefore $m^2 = 2e^2 + 1$; which is what ought to be the result from the consideration, that $a = e^2 + 1$, and $n = 2e$, which gives $an^2 + 1 = 4e^4 + 4e^2 + 1$, the square of $2e^2 + 1$. For example, let $a = 17$, so that $e = 4$, and we shall have $17n^2 + 1 = m^2$; by making $n = 8$, and $m = 33$.

111. Lastly, let $a = e^2 + 2$, or greater than a square by 2. Here, we have $(e^2 + 2)n^2 + 1 = m^2$, and, as before, $m > en$; therefore we shall suppose $m = en + p$, and shall thus have

$$e^2n^2 + 2n^2 + 1 = e^2n^2 + 2enp + p^2, \text{ or}$$
$$2n^2 = 2epn + p^2 - 1, \text{ which gives}$$

$$n = \frac{ep + \sqrt{(e^2p^2 + 2p^2 - 2)}}{2}.$$

Let $p = 1$, we shall find $n = e$, and $m = e^2 + 1$; and, in fact, since $a = e^2 + 2$, and $n = e$, we have $an^2 + 1 = e^4 + 2e^2 + 1$, which is the square of $e^2 + 1$.

For example, let $a = 11$, so that $e = 3$; we shall find $11n^2 + 1 = m^2$, by making $n = 3$, and $m = 10$. If we

* In this case, likewise, the radical sign vanishes, if we make $p = 0$: and this supposition incontestably gives the least possible numbers for $m$ and $n$, namely, $n = 1$, and $m = e$; that is to say, if $e = 5$, the formula $24n^2 + 1$ becomes a square by making $n = 1$; and the root of this square will be $m = e = 5$. F. T.

supposed $a = 83$, we should have $e = 9$, and

$$83n^2 + 1 = m^2, \text{ where } n = 9, \text{ and } m = 82\,*.$$

* Our author might have added here another very obvious case, which is when $a$ is of the form $e^2 \pm \frac{2}{c}e$; for then by making $n = c$, our formula $an^2 + 1$, becomes $e^2c^2 \pm 2ce + 1 = (ec \pm 1)^2$. I was led to the consideration of the above form, from having observed that the square roots of all numbers included in this formula are readily obtained by the method of continued fractions, the quotient figures, from which the fractions are derived, following a certain determined law, of two terms, readily observed, and that whenever this is the case, the method which is given above is also applied with great facility. And as a great many numbers are included in the above form, I have been induced to place it here, as a means of abridging the operations in those particular cases.

The reader is indebted to Mr. P. Barlow of the Royal Academy, Woolwich, for the above note; and also for a few more in this Second Part, which are distinguished by the signature, B.

TABLE, shewing for each value of $a$ the least numbers $m$ and $n$, that will give $m^2 = an^2 + 1$ *; or that will render $an^2 + 1$ a square.

| $a$ | $n$ | $m$ | $a$ | $n$ | $m$ |
|---|---|---|---|---|---|
| 2 | 2 | 3 | 53 | 9100 | 66249 |
| 3 | 1 | 2 | 54 | 66 | 485 |
| 5 | 4 | 9 | 55 | 12 | 89 |
| 6 | 2 | 5 | 56 | 2 | 15 |
| 7 | 3 | 8 | 57 | 20 | 151 |
| 8 | 1 | 3 | 58 | 2574 | 19603 |
| 10 | 6 | 19 | 59 | 69 | 530 |
| 11 | 3 | 10 | 60 | 4 | 31 |
| 12 | 2 | 7 | 61 | 226153980 | 1766319049 |
| 13 | 180 | 649 | 62 | 8 | 63 |
| 14 | 4 | 15 | 63 | 1 | 8 |
| 15 | 1 | 4 | 65 | 16 | 129 |
| 17 | 8 | 33 | 66 | 8 | 65 |
| 18 | 4 | 17 | 67 | 5967 | 48842 |
| 19 | 39 | 170 | 68 | 4 | 33 |
| 20 | 2 | 9 | 69 | 936 | 7775 |
| 21 | 12 | 55 | 70 | 30 | 251 |
| 22 | 42 | 197 | 71 | 413 | 3480 |
| 23 | 5 | 24 | 72 | 2 | 17 |
| 24 | 1 | 5 | 73 | 267000 | 2281249 |
| 26 | 10 | 51 | 74 | 430 | 3699 |
| 27 | 5 | 26 | 75 | 3 | 26 |
| 28 | 24 | 127 | 76 | 6630 | 57799 |
| 29 | 1820 | 9801 | 77 | 40 | 351 |
| 30 | 2 | 11 | 78 | 6 | 53 |
| 31 | 273 | 1520 | 79 | 9 | 80 |
| 32 | 3 | 17 | 80 | 1 | 9 |
| 33 | 4 | 23 | 82 | 18 | 163 |
| 34 | 6 | 35 | 83 | 9 | 82 |
| 35 | 1 | 6 | 84 | 6 | 55 |
| 37 | 12 | 73 | 85 | 30996 | 285769 |
| 38 | 6 | 37 | 86 | 1122 | 10405 |
| 39 | 4 | 25 | 87 | 3 | 28 |
| 40 | 3 | 19 | 88 | 21 | 197 |
| 41 | 320 | 2049 | 89 | 53000 | 500001 |
| 42 | 2 | 13 | 90 | 2 | 19 |
| 43 | 531 | 3482 | 91 | 165 | 1574 |
| 44 | 30 | 199 | 92 | 120 | 1151 |
| 45 | 24 | 161 | 93 | 1260 | 12151 |
| 46 | 3588 | 24335 | 94 | 221064 | 2143295 |
| 47 | 7 | 48 | 95 | 4 | 39 |
| 48 | 1 | 7 | 96 | 5 | 49 |
| 50 | 14 | 99 | 97 | 6377352 | 62809633 |
| 51 | 7 | 50 | 98 | 10 | 99 |
| 52 | 90 | 649 | 99 | 1 | 10 |

* See Article 8 of the additions by De la Grange.