numerous, it will be worth while to fix on some characters, by which their impossibility may be perceived, in order that we may be often saved the trouble of useless trials; which shall form the subject of the following chapter*.

CHAP. V.

Of the Cases *in which the* Formula $a + bx + cx^2$ *can never become a* Square.

63. As our general formula is composed of three terms, we shall observe, in the first place, that it may always be transformed into another, in which the middle term is wanting. This is done by supposing $x = \frac{y-b}{2c}$; which substitution changes the formula into

$a + \frac{by-b^2}{2c} + \frac{y^2-2by+b^2}{4c}$; or $\frac{4ac-b^2+y^2}{4c}$; and since this must be a square, let us make it equal to $\frac{z^2}{4}$, we shall then have $4ac - b^2 + y^2 = \frac{4cz^2}{4}, = cz^2$, and, consequently, $y^2 = cz^2 + b^2 - 4ac$. Whenever, therefore, our formula is a square, this last $cz^2 + b^2 - 4ac$ will be so likewise; and reciprocally, if this be a square, the proposed formula will be a square also. If therefore we write t, instead of $b^2 - 4ac$, the whole will be reduced to determining whether a quantity of the form $cz^2 + t$ can become a square or not. And as this formula consists only of two terms, it is certainly much easier to judge from that whether it be possible or not; but in any further inquiry we must be guided by the nature of the given numbers c and t.

64. It is evident that if $t = 0$, the formula cz^2 can become a square only when c is a square; for the quotient arising from the division of a square by another square being likewise a square, the quantity cz^2 cannot be a square, unless

* See the Appendix to this chapter, at Article 5. of the Additions by De la Grange. p. 537.

$\frac{cz^2}{z^2}$, that is to say, c, be one. So that when c is not a square, the formula cz^2 can by no means become a square; and on the contrary, if c be itself a square, cz^2 will also be a square, whatever number be assumed for z.

65. If we wish to consider other cases, we must have recourse to what has been already said on the subject of different kinds of numbers, considered with relation to their division by other numbers.

We have seen, for example, that the divisor 3 produces three different kinds of numbers. The first comprehends the numbers which are divisible by 3, and may be expressed by the formula $3n$.

The second kind comprehends the numbers which, being divided by 3, leave the remainder 1, and are contained in the formula $3n + 1$.

To the third class belong numbers which, being divided by 3, leave 2 for the remainder, and which may be represented by the general expression $3n + 2$.

Now, since all numbers are comprehended in these three formulæ, let us therefore consider their squares. First, if the question relate to a number included in the formula $3n$, we see that the square of this quantity being $9n^2$, it is divisible not only by 3, but also by 9.

If the given number be included in the formula $3n + 1$, we have the square $9n^2 + 6n + 1$, which, divided by 3, gives $3n^2 + 2n$, with the remainder 1; and which, consequently, belongs to the second class, $3n + 1$. Lastly, if the number in question be included in the formula $3n + 2$, we have to consider the square $9n^2 + 12n + 4$; and if we divide it by 3, we obtain $3n^2 + 4n + 1$, and the remainder 1; so that this square belongs, as well as the former, to the class $3n + 1$.

Hence it is obvious, that square numbers are only of two kinds with relation to the number 3; for they are either divisible by 3, and in this case are necessarily divisible also by 9; or they are not divisible by 3, in which case the remainder is always 1, and never 2; for which reason, no number contained in the formula $3n + 2$ can be a square.

66. It is easy, from what has just been said, to shew, that the formula $3x^2 + 2$ can never become a square, whatever integer, or fractional number, we choose to substitute for x. For, if x be an integer number, and we divide the formula $3x^2 + 2$ by 3, there remains 2; therefore it cannot be a

square. Next, if x be a fraction, let us express it by $\frac{t}{u}$, supposing it already reduced to its lowest terms, and that t and u have no common divisor. In order, therefore, that $\frac{3t^2}{u^2} + 2$ may be a square, we must obtain, after multiplying by u^2, $3t^2 + 2u^2$ also a square. Now, this is impossible; for the number u is either divisible by 3, or it is not: if it be, t will not be so, for t and u have no common divisor, since the fraction $\frac{t}{u}$ is in its lowest terms. Therefore, if we make $u = 3f$, as the formula becomes $3t^2 + 18f^2$, it is evident that it can be divided by 3 only once, and not twice, as it must necessarily be if it were a square; in fact, if we divide by 3, we obtain $t^2 + 6f^2$. Now, though one part, $6f^2$, is divisible by 3, yet the other, t^2, being divided by 3, leaves 1 for a remainder.

Let us now suppose that u is not divisible by 3, and see what results from that supposition. Since the first term is divisible by 3, we have only to learn what remainder the second term, $2u^2$, gives. Now, u^2 being divided by 3, leaves the remainder 1, that is to say, it is a number of the class $3n + 1$; so that $2u^2$ is a number of the class $6n + 2$; and dividing it by 3, the remainder is 2; consequently, the formula $3t^2 + 2u^2$, if divided by 3, leaves the remainder 2, and is certainly not a square number.

67. We may, in the same manner, demonstrate, that the formula $3t^2 + 5u^2$, likewise can never become a square, nor any one of the following:

$$3t^2 + 8u^2,\ 3t^2 + 11u^2,\ 3t^2 + 14u^2, \text{ \&c.}$$

in which the numbers 5, 8, 11, 14, &c. divided by 3, leave 2 for a remainder. For, if we suppose that u is divisible by 3, and, consequently, that t is not so, and if we make $u = 3n$, we shall always be brought to formulæ divisible by 3, but not divisible by 9: and if u were not divisible by 3, and consequently u^2 a number of the kind $3n + 1$, we should have the first term, $3t^2$, divisible by 3, while the second terms, $5u^2$, $8u^2$, $11u^2$, &c. would have the forms $15n + 5$, $24n + 8$, $33n + 11$, &c. and, when divided by 3, would constantly leave the remainder 2.

68. It is evident that this remark extends also to the general formula, $3t^2 + (3n + 2) \times u^2$, which can never become a square, even by taking negative numbers for n. If, for example, we should make $n = -1$, I say, it is im-

possible for the formula $3t^2 - u^2$ to become a square. This is evident, if u be divisible by 3: and if it be not, then u^2 is a number of the kind $3n + 1$, and our formula becomes $3t^2 - 3n - 1$, which, being divided by 3, gives the remainder $- 1$, or $+ 2$; and in general, if n be $= - m$, we obtain the formula $3t^2 - (3m - 2)u^2$, which can never become a square.

69. So far, therefore, are we led by considering the divisor 3; if we now consider 4 also as a divisor, we see that every number may be comprised in one of the four following formulæ;

$$4n,\ 4n + 1,\ 4n + 2,\ 4n + 3.$$

The square of the first of these classes of numbers is $16n^2$; and, consequently, it is divisible by 16.

That of the second class, $4n + 1$, is $16n^2 + 8n + 1$; which if divided by 8, the remainder is 1; so that it belongs to the formula $8n + 1$.

The square of the third class, $4n + 2$, is $16n^2 + 16n + 4$; which if we divide by 16, there remains 4; therefore this square is included in the formula $16n + 4$.

Lastly, the square of the fourth class, $4n + 3$, being $16n^2 + 24n + 9$, it is evident that dividing by 8 there remains 1.

70. This teaches us, in the first place, that all the even square numbers are either of the form $16n$, or $16n + 4$; and, consequently, that all the other even formulæ, namely,

$$16n+2,\ 16n+6,\ 16n+8,\ 16n+10,\ 16n+12,\ 16n+14,$$

can never become square numbers.

Secondly, that all the odd squares are contained in the formula $8n + 1$; that is to say, if we divide them by 8, they leave a remainder of 1. And hence it follows, that all the other odd numbers, which have the form either of $8n + 3$, or of $8n + 5$, or of $8n + 7$, can never be squares.

71. These principles furnish a new proof, that the formula $3t^2 + 2u^2$ cannot be a square. For, either the two numbers t and u are both odd, or the one is even and the other odd. They cannot be both even, because in that case they would, at least, have the common divisor 2. In the first case, therefore, in which both t^2 and u^2 are contained in the formula $8n + 1$, the first term $3t^2$, being divided by 8, would leave the remainder 3, and the other term $2u^2$ would leave the remainder 2; so that the whole remainder would be 5: consequently, the formula in question cannot be a square. But, if the second case be supposed, and t be even, and u odd, the first term $3t^2$ will be divisible by 4, and the

second term $2u^2$, if divided by 4, will leave the remainder 2; so that the two terms together, when divided by 4, leave a remainder of 2, and therefore cannot form a square. Lastly, if we were to suppose u an even number, as $2s$, and t odd, so that t^2 is of the form $8n + 1$, our formula would be changed into this, $24n + 3 + 8s^2$; which, divided by 8, leaves 3, and therefore cannot be a square.

This demonstration extends to the formula $3t^2 + (8n + 2)u^2$; also to this, $(8m + 3)t^2 + 2u^2$, and even to this, $(8m + 3)t^2 + (8n + 2)u^2$; in which we may substitute for m and n all integer numbers, whether positive or negative.

72. But let us proceed farther, and consider the divisor 5, with respect to which all numbers may be ranged under the five following classes:

$$5n,\ 5n + 1,\ 5n + 2,\ 5n + 3,\ 5n + 4.$$

We remark, in the first place, that if a number be of the first class, its square will have the form $25n^2$; and will consequently be divisible not only by 5, but also by 25.

Every number of the second class will have a square of the form $25n^2 + 10n + 1$; and as dividing by 5 gives the remainder 1, this square will be contained in the formula $5n + 1$.

The numbers of the third class will have for their square $25n^2 + 20n + 4$; which, divided by 5, gives 4 for the remainder.

The square of a number of the fourth class is $25n^2 + 30n + 9$; and if it be divided by 5, there remains 4.

Lastly, the square of a number of the fifth class is $25n^2 + 40n + 16$; and if we divide this square by 5, there will remain 1.

When a square number therefore cannot be divided by 5, the remainder after division will always be 1, or 4, and never 2, or 3: hence it follows, that no square number can be contained in the formula $5n + 2$, or $5n + 3$.

73. From this it may be proved, that neither the formula $5t^2 + 2u^2$, nor $5t^2 + 3u^2$, can be a square. For, either u is divisible by 5, or it is not: in the first case, these formulæ will be divisible by 5, but not by 25; therefore they cannot be squares. On the other hand, if u be not divisible by 5, u^2 will either be of the form $5n + 1$, or $5n + 4$. In the first of these cases, the formula $5t^2 + 2u^2$ becomes $5t^2 + 10n + 2$; which, divided by 5, leaves a remainder of 2; and the formula $5t^2 + 3u^2$ becomes $5t^2 + 15n + 3$; which, being divided by 5, gives a remainder of 3; so that neither the one nor the other can be a square. With regard to the case of $u^2 = 5n + 4$, the first formula becomes $5t^2 + 10n + 8$;

which, divided by 5, leaves 3; and the other becomes $5t^2 + 15n + 12$, which, divided by 5, leaves 2; so that in this case also, neither of the two formulæ can be a square.

For a similar reason, we may remark, that neither the formula $5t^2 + (5n + 2)u^2$, nor $5t^2 + (5n + 3)u^2$, can become a square, since they leave the same remainders that we have just found. We might even in the first term write $5mt^2$, instead of $5t^2$, provided m be not divisible by 5.

74. Since all the even squares are contained in the formula $4n$, and all the odd squares in the formula $4n + 1$; and, consequently, since neither $4n + 2$, nor $4n + 3$, can become a square, it follows that the general formula $(4m + 3)t^2 + (4n + 3)u^2$ can never be a square. For if t be even, t^2 will be divisible by 4, and the other term, being divided by 4, will give 3 for a remainder; and, if we suppose the two numbers t and u odd, the remainders of t^2 and of u^2 will be 1; consequently, the remainder of the whole formula will be 2: now, there is no square number, which, when divided by 4, leaves a remainder of 2.

We shall remark, also, that both m and n may be taken negatively, or $= 0$, and still the formulæ $3t^2 + 3u^2$, and $3t^2 - u^2$, cannot be transformed into squares.

75. In the same manner as we have found for a few divisors, that some kinds of numbers can never become squares, we might determine similar kinds of numbers for all other divisors.

If we take the divisor 7, we shall have to distinguish seven different kinds of numbers, the squares of which we shall also examine.

Kinds of numbers.		Their squares are of the kind,	
1.	$7n$	$49n^2$	$7n$
2.	$7n + 1$	$49n^2 + 14n + 1$	$7n + 1$
3.	$7n + 2$	$49n^2 + 28n + 4$	$7n + 4$
4.	$7n + 3$	$49n^2 + 42n + 9$	$7n + 2$
5.	$7n + 4$	$49n^2 + 56n + 16$	$7n + 2$
6.	$7n + 5$	$49n^2 + 70n + 25$	$7n + 4$
7.	$7u + 6$	$49n^2 + 84n + 36$	$7n + 1.$

Therefore, since the squares which are not divisible by 7, are all contained in the three formulæ $7n + 1$, $7n + 2$, $7n + 4$, it is evident, that the three other formulæ, $7n + 3$, $7n + 5$, and $7n + 6$, do not agree with the nature of squares.

76. To make this conclusion still more apparent, we shall remark, that the last kind, $7n + 6$, may be also expressed

by $7n - 1$; that, in the same manner, the formula $7n + 5$ is the same as $7n - 2$, and $7n + 4$ the same as $7n - 3$. This being the case, it is evident, that the squares of the two classes of numbers, $7n + 1$, and $7n - 1$, if divided by 7, will give the same remainder 1; and that the squares of the two classes, $7n + 2$, and $7n - 2$, ought to resemble each other in the same respect, each leaving the remainder 4.

77. In general, therefore, let the divisor be any number whatever, which we shall represent by the letter d, the different classes of numbers which result from it will be

$$dn;$$
$$dn + 1,\ dn + 2,\ dn + 3, \text{ \&c.}$$
$$dn - 1,\ dn - 2,\ dn - 3, \text{ \&c.}$$

in which the squares of $dn + 1$, and $dn - 1$, have this in common, that, when divided by d, they leave the remainder 1, so that they belong to the same formula, $dn + 1$; in the same manner, the squares of the two classes $dn + 2$, and $dn - 2$, belong to the same formula, $dn + 4$. So that we may conclude, generally, that the squares of the two kinds, $dn + a$, and $dn - a$, when divided by d, give a common remainder a^2, or that which remains in dividing a^2 by d.

78. These observations are sufficient to point out an infinite number of formulæ, such as $at^2 + bu^2$, which cannot by any means become squares. Thus, by considering the divisor 7, it is easy to perceive, that none of these three formulæ, $7t^2 + 3u^2$, $7t^2 + 5u^2$, $7t^2 + 6u^2$, can ever become a square; because the division of u^2 by 7 only gives the remainders 1, 2, or 4; and, in the first of these formulæ, there remains either 3, or 6, or 5; in the second, 5, 3, or 6; and in the third, 6, 5, or 3; which cannot take place in square numbers. Whenever, therefore, we meet with such formulæ, we are certain that it is useless to attempt discovering any case, in which they can become squares: and, for this reason, the considerations, into which we have just entered, are of some importance.

If, on the other hand, the formula proposed is not of this nature, we have seen in the last chapter, that it is sufficient to find a single case, in which it becomes a square, to enable us to deduce from it an infinite number of similar cases.

The given formula, Art. 63, was properly $ax^2 + b$; and, as we usually obtain fractions for x, we supposed $x = \frac{t}{u}$, so that the problem, in reality, is to transform $at^2 + bu^2$ into a square.

But there is frequently an infinite number of cases, in which x may be assigned even in integer numbers; and the determination of those cases shall form the subject of the following chapter.

CHAP. VI.

Of the Cases *in* Integer Numbers, *in which the* Formula $ax^2 + b$ *becomes a* Square.

79. We have already shewn, Art. 63, how such formulæ as $a + bx + cx^2$, are to be transformed, in order that the second term may be destroyed; we shall therefore confine our present inquiries to the formula $ax^2 + b$, in which it is required to find for x only integer numbers, which may transform that formula into a square. Now, first of all, such a formula must be possible; for, if it be not, we shall not even obtain fractional values of x, far less integer ones.

80. Let us suppose then $ax^2 + b = y^2$; a and b being integer numbers, as well as x and y.

Now, here it is absolutely necessary for us to know, or to have already found a case in integer numbers; otherwise it would be lost labor to seek for other similar cases, as the formula might happen to be impossible.

We shall, therefore, suppose that this formula becomes a square, by making $x = f$, and we shall represent that square by g^2, so that $af^2 + b = g^2$, where f and g are known numbers. Then we have only to deduce from this case other similar cases; and this inquiry is so much the more important, as it is subject to considerable difficulties; which, however, we shall be able to surmount by particular artifices.

81. Since we have already found $af^2 + b = g^2$, and likewise, by hypothesis, $ax^2 + b = y^2$, let us subtract the first equation from the second, and we shall obtain a new one, $ax^2 - af^2 = y^2 - g^2$, which may be represented by factors in the following manner; $a(x + f) \times (x - f) = (y + g) \times (y - g)$, and which, by multiplying both sides by pq, becomes $apq(x + f) \times (x - f) = pq(y + g) \times (y - g)$. If we now decompound this equation, by making $ap(x + f) = q(y + g)$, and $q(x - f) = p(y - g$, we may derive, from these two equations, values of the two letters x and y. The