Therefore $r^4 = \frac{81}{16}s^4 + \frac{27}{2}s^3t + \frac{27}{2}s^2t^2 + 6st^3 + t^4$
$+ 4r^3s = \frac{27}{2}s^4 + 27s^3t + 18s^2t^2 + 4st^3$
$- 6r^2s^2 = - \frac{27}{2}s^4 - 18s^3t - 6s^2t^2$
$- 4rs^3 = - 6s^4 - 4s^3t$
$+ s^4 = + s^4$; and, consequently, the formula will be $\frac{1}{16}s^4 + \frac{37}{2}s^3t + \frac{51}{2}s^2t^2 + 10st^3 + t^4$.

This formula ought also to be a square, if multiplied by 16, by which means it becomes

$$s^4 + 296s^3t + 408s^2t^2 + 160st^3 + 16t^4.$$

Let us make this equal to the square of $s^2 + 148st - 4t^2$, that is, to $s^4 + 296s^3t + 21896s^2t^2 - 1184st^3 + 16t^4$; the first two terms, and the last, are destroyed on both sides, and we thus obtain the equation

$$21896s - 1184t = 408s + 160t, \text{ which gives}$$

$$\frac{s}{t} = \frac{1344}{21488} = \frac{336}{5372} = \frac{84}{1343}.$$

Therefore, since $s = 84$, and $t = 1343$, we shall have $r = \frac{3}{2}s + t = 1469$, and, consequently,

$$x = r^4 - 6r^2s^2 + s^4 = 4565486027761, \text{ and}$$
$$y = 4r^3s - 4rs^3 = 1061652293520.$$

CHAP. XV.

Solutions *of some* Questions, *in which* Cubes *are required.*

241. In the preceding chapter, we have considered some questions, in which it was required to transform certain formulæ into squares, and they afforded an opportunity of explaining several artifices requisite in the application of the rules which have been given. It now remains, to consider questions, which relate to the transformation of certain formulæ into cubes; and the following solutions will throw some light on the rules, which have been already explained for transformations of this kind.

242. *Question* 1. It is required to find two cubes, x^3, and y^3, whose sum may be a cube.

Since $x^3 + y^3$ must be a cube, if we divide this formula by y^3, the quotient ought likewise to be a cube, or $\frac{x^3}{y^3} + 1 = c$. If, therefore, $\frac{x}{y} = z - 1$, we shall have

$z^3 - 3z^2 + 3z - 1^3 = c$. If we should here, according to the rules already given, suppose the cube root to be $z-u$, and, by comparing the formula with the cube $z^3 - 3uz^2 + 3u^2z - u^3$, determine u so, that the second term may also vanish, we should have $u = 1$; and the other terms forming the equation $3z = 3u^2z - u^3 = 3z - 1$, we should find $z = \infty$, from which we can draw no conclusion. Let us therefore rather leave u undetermined, and deduce z from the quadratic equation $-3z^2 + 3z = -3uz^2 + 3u^2z - u^3$, or $3uz^2 - 3z^2 = 3u^2z - 3z - u^3$, or $3(u-1)z^2 = 3(u^2-1)z - u^3$, or

$z^2 = (u+1)z - \dfrac{u^3}{3(u-1)}$; from this we shall find

$$z = \frac{u+1}{2} \pm \sqrt{\left(\frac{u^2+2u+1}{4} - \frac{u^3}{3(u-1)}\right)}$$

or $z = \dfrac{u+1}{2} \mp \sqrt{\left(\dfrac{-u^3+3u^2-3u-3}{12(u-1)}\right)}$; so that the question is reduced to transforming the fraction under the radical sign into a square. For this purpose, let us first multiply the two terms by $3(u-1)$, in order that the denominator becoming a square, namely, $36(u-1)^2$, we may only have to consider the numerator $-3u^4 + 12u^3 - 18u^2 + 9$: and, as the last term is a square, we shall suppose the formula, according to the rule, equal to the square of $gu^2 + fu + 3$, that is, to $g^2u^4 + 2fgu^3 + f^2u^2 + 6gu^2 + 6fu + 9$. We may make the last three terms disappear, by putting $6f = 0$, or $f = 0$, and $6g + f^2 = -18$, or $g = -3$; and the remaining equation, namely,

$$-3u + 12 = g^2u + 2fu = 9u,$$

will give $u = 1$. But from this value we learn nothing; so that we shall proceed by writing $u = 1 + t$. Now, as our formula becomes in this case $-12t - 3t^4$, which cannot be a square, unless t be negative, let us at once make $t = -s$; by these means we have the formula $12s - 3s^4$, which becomes a square in the case of $s = 1$. But here we are stopped again; for when $s = 1$, we have $t = -1$, and $u = 0$, from which we can draw no conclusion, except that in whatever manner we set about it, we shall never find a value that will bring us to the end proposed; and hence we may already infer, with some degree of certainty, that it is impossible to find two cubes whose sum is a cube. But we shall be fully convinced of this from the following demonstration.

243. *Theorem.* It is impossible to find any two cubes, whose sum, or difference, is a cube.

We shall begin by observing, that if this impossibility applies to the sum, it applies also to the difference, of two cubes. In fact, if it be impossible for $x^3 + y^3 = z^3$, it is also impossible for $z^3 - y^3 = x^3$. Now, $z^3 - y^3$ is the difference of two cubes; therefore, if the one be possible, the other is so likewise. This being laid down, it will be sufficient, if we demonstrate the impossibility either in the case of the sum, or difference; which demonstration requires the following chain of reasoning.

1. We may consider the numbers x and y as prime to each other; for if they had a common divisor, the cubes would also be divisible by the cube of that divisor. For example, let $x = ma$, and $y = mb$, we shall then have $x^3 + y^3 = m^3a^3 + m^3b^3$; now if this formula be a cube, $a^3 + b^3$ is a cube also.

2. Since, therefore, x and y have no common factor, these two numbers are either both odd, or the one is even and the other odd. In the first case, z would be even, and in the other that number would be odd. Consequently, of these three numbers x, y, and z, there is always one which is even, and two that are odd; and it will therefore be sufficient for our demonstration to consider the case in which x and y are both odd: because we may prove the impossibility in question either for the sum, or for the difference; and the sum only happens to become the difference, when one of the roots is negative.

3. If therefore x and y are odd, it is evident that both their sum and their difference will be an even number. Therefore let $\frac{x+y}{2} = p$, and $\frac{x-y}{2} = q$, and we shall have $x = p + q$, and $y = p - q$; whence it follows, that one of the two numbers, p and q, must be even, and the other odd. Now, we have, by adding $(p + q)^3 = x^3$, to $(p - q)^3 = y^3$, $x^3 + y^3 = 2p^3 + 6pq^2 = 2p(p^2 + 3q^2)$; so that it is required to prove that this product $2p(p^2+3q^2)$ cannot become a cube; and if the demonstration were applied to the difference, we should have $x^3 - y^3 = 6p^2q + 2q^3 = 2q(q^2 + 3p^2)$, a formula precisely the same as the former, if we substitute p and q for each other. Consequently, it is sufficient for our purpose to demonstrate the impossibility of the formula $2p(p^2 + 3q^2)$, since it will necessarily follow, that neither the sum nor the difference of two cubes can become a cube.

4. If therefore $2p(p^2 + 3q^2)$ were a cube, that cube would be even, and, consequently, divisible by 8: con-

sequently, the eighth part of our formula, or $\frac{1}{4}p(p^2 + 3q^2)$, would necessarily be a whole number, and also a cube. Now, we know that one of the numbers p and q is even, and the other odd; so that $p^2 + 3q^2$ must be an odd number, which not being divisible by 4, p must be so, or $\frac{p}{4}$ must be a whole number.

5. But in order that the product $\frac{1}{4}p(p^2 + 3q^2)$ may be a cube, each of these factors, unless they have a common divisor, must separately be a cube; for if a product of two factors, that are prime to each other, be a cube, each of itself must necessarily be a cube; and if these factors have a common divisor, the case is different, and requires a particular consideration. So that the question here is, to know if the factors p, and $p^2 + 3q^2$, might not have a common divisor. To determine this, it must be considered, that if these factors have a common divisor, the numbers p^2, and $p^2 + 3q^2$, will have the same divisor; that the difference also of these numbers, which is $3q^2$, will have the same common divisor with p^2; and that, since p and q are prime to each other, these numbers p^2, and $3q^2$, can have no other common divisor than 3, which is the case when p is divisible by 3.

6. We have consequently two cases to examine: the one is, that in which the factors p, and $p^2 + 3q^2$, have no common divisor, which happens always, when p is not divisible by 3; the other case is, when these factors have a common divisor, and that is when p may be divided by 3; because then the two numbers are divisible by 3. We must carefully distinguish these two cases from each other, because each requires a particular demonstration.

7. *Case* 1. Suppose that p is not divisible by 3, and, consequently, that our two factors $\frac{p}{4}$, and $p^2 + 3q^2$, are prime to each other; so that each must separately be a cube. Now, in order that $p^2 + 3q^2$ may become a cube, we have only, as we have seen before, to suppose
$p+q\sqrt{-3}=(t+u\sqrt{-3})^3$, and $p-q\sqrt{-3}=(t-u\sqrt{-3})^3$, which gives $p^2 + 3q^2 = (t^2 + 3u^2)^3$, which is a cube, and gives us $p = t^3 - 9tu^2 = t(t^2 - 9u^2)$, also
$q = 3t^2u - 3u^3 = 3u(t^2 - u^2)$. Since therefore q is an odd number, u must also be odd; and, consequently, t must be even, because otherwise $t^2 - u^2$ would be even.

8. Having transformed $p^2 + 3q^2$ into a cube, and having

found $p = t(t^2 - 9u^2) = t(t + 3u) \times (t - 3u)$, it is also required that $\frac{p}{4}$, and consequently $2p$, be a cube; or, which comes to the same, that the formula $2t(t + 3u) \times (t - 3u)$ be a cube. But here it must be observed that t is an even number, and not divisible by 3; since otherwise p would be divisible by 3, which we have expressly supposed not to be the case: so that the three factors, $2t$, $t + 3u$, and $t - 3u$, are prime to each other; and each of them must separately be a cube. If, therefore, we make $t + 3u = f^3$, and $t - 3u = g^3$, we shall have $2t = f^3 + g^3$. So that, if $2t$ is a cube, we shall have two cubes f^3, and g^3, whose sum would be a cube, and which would evidently be much less than the cubes x^3 and y^3 assumed at first; for as we first made $x = p + q$, and $y = p - q$, and have now determined p and q by the letters t and u, the numbers x and y must necessarily be much greater than t and u.

9. If, therefore, there could be found in great numbers two such cubes as we require, we should also be able to assign in less numbers two cubes whose sum would make a cube, and in the same manner we should be led to cubes always less. Now, as it is very certain that there are no such cubes among small numbers, it follows that there are not any among the greater numbers. This conclusion is confirmed by that which the second case furnishes, and which will be seen to be the same.

10. *Case* 2. Let us now suppose, that p is divisible by 3, and that q is not so, and let us make $p = 3r$; our formula will then become $\frac{3r}{4} \times (9r^2 + 3q^2)$, or $\frac{9}{4}r(3r^2 + q^2)$; and these two factors are prime to each other, since $3r^2 + q^2$ is neither divisible by 2 nor by 3, and r must be even as well as p; therefore each of these two factors must separately be a cube.

11. Now, by transforming the second factor $3r^2 + q^2$, or $q^2 + 3r^2$, we find, in the same manner as before, $q = t(t^2 - 9u^2)$, and $r = 3u(t^2 - u^2)$; and it must be observed, that since q was odd, t must be here likewise an odd number, and u must be even.

12. But $\frac{9r}{4}$ must also be a cube; or multiplying by the cube $\frac{8}{27}$, we must have $\frac{2r}{3}$, or

$2u(t^2 - u^2) = 2u(t + u) \times (t - u)$ a cube; and as these three factors are prime to each other, each must of itself be a cube. Suppose therefore $t + u = f^3$, and $t - u = g^3$, we shall have $2u = f^3 - g^3$; that is to say, if $2u$ were a cube, $f^3 - g^3$ would be a cube. We should consequently have two cubes, f^3 and g^3, much smaller than the first, whose difference would be a cube, and that would enable us also to find two cubes whose sum would be a cube; since we should only have to make $f^3 - g^3 = h^3$, in order to have $f^3 = h^3 + g^3$, or a cube equal to the sum of two cubes. Thus, the foregoing conclusion is fully confirmed; for as we cannot assign, in great numbers, two cubes whose sum or difference is a cube, it follows from what has been before observed, that no such cubes are to be found among small numbers.

244. Since it is impossible, therefore, to find two cubes, whose sum or difference is a cube, our first question falls to the ground: and, indeed, it is more usual to enter on this subject with the question of determining three cubes, whose sum may make a cube; supposing, however, two of those cubes to be arbitrary, so that it is only required to find the third. We shall therefore proceed immediately to this question.

245. *Question* 2. Two cubes a^3, and b^3, being given, required a third cube, such, that the three cubes added together may make a cube.

It is here required to transform into a cube the formula $a^3 + b^3 + x^3$; which cannot be done unless we already know a satisfactory case; but such a case occurs immediately; namely, that of $x = -a$. If therefore we make $x = y - a$, we shall have $x^3 = y^3 - 3ay^2 + 3a^2y - a^3$; and, consequently, it is the formula $y^3 - 3ay^2 + 3a^2y + b^3$ that must become a cube. Now, the first and the last term here being cubes, we immediately find two solutions.

1. The first requires us to represent the root of the formula by $y + b$, the cube of which is $y^3 + 3by^2 + 3b^2y + b^3$; and we thus obtain $-3ay + 3a^2 = 3by + 3b^2$; and, consequently, $y = \frac{a^2 - b^2}{a + b} = a - b$; but $x = -b$, so that this solution is of no use.

2. But we may also represent the root by $fy + b$, the cube of which is $f^3y^3 + 3bf^2y^2 + 3b^2fy + b^3$, and then determine f in such a manner, that the third terms may be destroyed, namely, by making $3a^2 = 3b^2f$, or $f = \frac{a^2}{b^2}$; for

we thus arrive at the equation

$$y - 3a = f^3y + 3bf^2 = \frac{a^6y}{b^6} + \frac{3a^4}{b^3},$$ which multiplied by b^6, becomes $b^6y - 3ab^6 = a^6y + 3a^4b^3$. This gives

$$y = \frac{3a^4b^3+3ab^6}{b^6-a^6} = \frac{3ab^3(a^3+b^3)}{b^6-a^6} = \frac{3ab^3}{b^3-a^3};$$ and, consequently,

$$x = y - a = \frac{2ab^3+a^4}{b^3-a^3} = a \times \frac{2b^3+a^3}{b^3-a^3}.$$ So that the two cubes a^3 and b^3 being given, we know also the root of the third cube sought; and if we would have that root positive, we have only to suppose b^3 to be greater than a^3. Let us apply this to some examples.

1. Let 1 and 8 be the two given cubes, so that $a = 1$, and $b = 2$; the formula $9 + x^3$ will become a cube, if $x = \frac{17}{7}$; for we shall have $9 + x^3 = \frac{8000}{343} = (\frac{20}{7})^3$.

2. Let the given cubes be 8 and 27, so that $a = 2$, and $b = 3$; the formula $35 + x^3$ will be a cube, when $x = \frac{124}{19}$.

3. If 27 and 64 be the given cubes, that is, if $a = 3$, and $b = 4$, the formula $91 + x^3$ will become a cube, if $x = \frac{465}{37}$.

And, generally, in order to determine third cubes for any two given cubes, we must proceed by substituting

$$\frac{2ab^3+a^4}{b^3-a^3} + z$$ instead of x, in the formula $a^3 + b^3 + x^3$;

for by these means we shall arrive at a formula like the preceding, which would then furnish new values of z; but it is evident that this would lead to very prolix calculations.

246. In this question, there likewise occurs a remarkable case; namely, that in which the two given cubes are equal, or $a = b$; for then we have $x = \frac{3a^4}{0} = \infty$; that is, we have no solution; and this is the reason why we are not able to resolve the problem of transforming into a cube the formula $2a^3 + x^3$. For example, let $a = 1$, or let this formula be $2 + x^3$, we shall find that whatever forms we give it, it will always be to no purpose, and we shall seek in vain for a satisfactory value of x. Hence, we may conclude with sufficient certainty, that it is impossible to find a cube equal to the sum of a cube, and of a double cube; or that the equation $2a^3 + x^3 = y^3$ is impossible. As this equation

gives $2a^3 = y^3 - x^3$, it is likewise impossible to find two cubes having their difference equal to the double of another cube; and the same impossibility extends to the sum of two cubes, as is evident from the following demonstration.

247. *Theorem.* Neither the sum nor the difference of two cubes can become equal to the double of another cube; or, in other words, the formula $x^3 \pm y^3 = 2z^3$ is always impossible, except in the evident case of $y = x$.

We may here also consider x and y as prime to each other; for if these numbers had a common divisor, it would be necessary for z to have the same divisor; and, consequently, for the whole equation to be divisible by the cube of that divisor. This being laid down, as $x^3 \pm y^3$ must be an even number, the numbers x and y must both be odd, in consequence of which both their sum and their difference must be even. Making, therefore, $\frac{x+y}{2} = p$, and $\frac{x-y}{2} = q$, we shall have $x = p + q$ and $y = p - q$; and of the two numbers p and q, the one must be even and the other odd. Now, from this, we obtain

$$x^3 + y^3 = 2p^3 + 6pq^2 = 2p(p^2 + 3q^2),$$
$$\text{and } x^3 - y^3 = 6p^2q + 2q^3 = 2q(3p^2 + q^2),$$

which are two formulæ perfectly similar. It will therefore be sufficient to prove that the formula $2p(p^2 + 3q^2)$ cannot become the double of a cube, or that $p(p^2 + 3q^2)$ cannot become a cube: which may be demonstrated in the following manner.

1. Two different cases again present themselves to our consideration: the one, in which the two factors p, and $p^2 + 3q^2$, have no common divisor, and must separately be a cube; the other in which these factors have a common divisor, which divisor, however, as we have seen (Art. 243), can be no other than 3.

2. *Case* 1. Supposing, therefore, that p is not divisible by 3, and that thus the two factors are prime to each other, we shall first reduce $p^2 + 3q^2$ to a cube by making $p = t(t^2 - 9u^2)$, and $q = 3u(t^2 - 9u^2)$; by which means it will only be farther necessary for p to become a cube. Now, t not being divisible by 3, since otherwise p would also be divisible by 3, the two factors t, and $t^2 - 9u^2$, are prime to one another, and, consequently, each must separately be a cube.

3. But the last factor has also two factors, namely $t + 3u$, and $t - 3u$, which are prime to each other, first because t is not divisible by 3, and, in the second place, because one of

the numbers t or u is even, and the other odd; for if these numbers were both odd, not only p, but also q, must be odd, which cannot be: therefore, each of these two factors, $t+3u$, and $t-3u$, must separately be a cube.

4. Therefore let $t+3u=f^3$, and $t-3u=g^3$, and we shall then have $2t=f^3+g^3$. Now, t must be a cube, which we shall denote by h^3, by which means we must have $f^3+g^3=2h^3$; consequently, we should have two cubes much smaller, namely, f^3 and g^3, whose sum would be the double of a cube.

5. *Case* 2. Let us now suppose p divisible by 3, and, consequently, that q is not so.

If we make $p=3r$, our formula becomes $3r(9r^2+3q^2)=9r(3r^2+q^2)$, and these factors being now numbers prime to one another, each must separately be a cube.

6. In order therefore to transform the second q^2+3r^2, into a cube, we shall make $q=t(t^2-9u^2)$, and $r=3u(t^2-u^2)$; and again one of the numbers t and u must be odd, and the other even, since otherwise the two numbers q and r would be even. Now, from this we obtain the first factor $9r=27u(t^2-u^2)$; and as it must be a cube, let us divide it by 27, and the formula $u(t^2-u^2)$, or $u(t+u)\times(t-u)$, must be a cube.

7. But these three factors being prime to each other, they must all be cubes of themselves. Let us therefore suppose for the last two $t+u=f^3$, and $t-u=g^3$, we shall then have $2u=f^3-g^3$; but as u must be a cube, we should in this way have two cubes, in much smaller numbers, whose difference would be equal to the double of another cube.

8. Since therefore we cannot assign, in small numbers, any cubes, whose sum or difference is the double of a cube, it is evident that there are no such cubes, even among the greatest numbers.

9. It will perhaps be objected, that our conclusion might lead to error; because there does exist a satisfactory case among these small numbers; namely, that of $f=g$. But it must be considered that when $f=g$, we have, in the first case, $t+3u=t-3u$, and therefore $u=0$; consequently, also $q=0$; and, as we have supposed $x=p+q$, and $y=p-q$, the first two cubes, x^3 and y^3, must have already been equal to one another, which case was expressly excepted. Likewise, in the second case, if $f=g$, we must have $t+u=t-u$, and also $u=0$: therefore $r=0$, and $p=0$; so that the first two cubes, x^3 and y^3, would again

become equal, which does not enter into the subject of the problem.

248. *Question* 3. Required in general three cubes, x^3, y^3, and z^3, whose sum may be equal to a cube.

We have seen that two of these cubes may be supposed to be known, and that from them we may determine the third, provided the two are not equal; but the preceding method furnishes in each case only one value for the third cube, and it would be difficult to deduce from it any new ones.

We shall now, therefore, consider the three cubes as unknown; and, in order to give a general solution, let us make $x^3 + y^3 + z^3 = v^3$. Here, by transposing one of the terms, we have $x^3 + y^3 = v^3 - z^3$, the conditions of which equation we may satisfy in the following manner.

1. Let $x = p + q$, and $y = p - q$, and we shall have, as before, $x^3 + y^3 = 2p(p^2 + 3q^2)$. Also, let $v = r + s$, and $z = r - s$, which gives $v^3 - z^3 = 2s(s^2 + 3r^2)$; therefore we must have $2p(p^2 + 3q^2) = 2s(s^2 + 3r^2)$, or

$$p(p^2 + 3q^2) = s(s^2 + 3r^2).$$

2. We have already seen (Art. 176), that a number, such as $p^2 + 3q^2$, can have no divisors except numbers of the same form. Since, therefore, these two formulæ, $p^2 + 3q^2$, and $s^2 + 3r^2$, must necessarily have a common divisor, let that divisor be $t^2 + 3u^2$.

3. And let us, therefore, make

$$p^2 + 3q^2 = (f^2 + 3g^2) \times (t^2 + 3u^2), \text{ and}$$
$$s^2 + 3r^2 = (h^2 + 3k^2) \times (t^2 + 3u^2),$$

and we shall have $p = ft + 3gu$, and $q = gt - fu$; consequently, $p^2 = f^2t^2 + 6fgtu + 9g^2u^2$, and

$$q^2 = g^2t^2 - 2fgtu + f^2u^2; \text{ whence,}$$
$$p^2 + 3q^2 = (f^2 + 3g^2)t^2 + (3f^2 + 9g^2)u^2; \text{ or}$$
$$p^2 + 3q^2 = (f^2 + 3g^2) \times (t^2 + 3u^2).$$

4. In the same manner, we may deduce from the other formula, $s = ht + 3ku$, and $r = kt - hu$; whence results the equation,

$$(ft + 3gu) \times (f^2 + 3g^2) \times (t^2 + 3u^2) =$$
$$(ht + 3ku) \times (h^2 + 3k^2) \times (t^2 + 3u^2),$$

which being divided by $t^2 + 3u^2$, and reduced, gives

$$ft(f^2 + 3g^2) + 3gu(f^2 + 3g^2) =$$
$$ht(h^2 + 3k^2) + 3ku(h^2 + 3k^2), \text{ or}$$
$$ft(f^2 + 3g^2) - ht(h^2 + 3k^2) =$$
$$3ku(h^2 + 3k^2) - 3gu(f^2 + 3g^2),$$

by which means $t = \dfrac{3k(h^2 + 3k^2) - 3g(f^2 + 3g^2)}{f(f^2 + 3g^2) - h(h^2 + 3k^2)}u.$

5. Let us now remove the fractions, by making

$$u = f(f^2 + 3g^2) - h(h^2 + 3k^2); \text{ then}$$
$$t = 3k(h^2 + 3k^2) - 3g(f^2 + 3g^2),$$

where we may give any values whatever to the letters f, g, h, and k.

6. When therefore we have determined, from these four numbers, the values of t and u, we shall have

$$p = ft + 3gu, \qquad q = gt - fu,$$
$$r = kt - hu, \qquad s = ht + 3ku;$$

whence we shall at last arrive at the solution of the question, $x = p + q$, $y = p - q$, $z = r - s$, and $v = r + s$; and this solution is general, so far as to comprehend all the possible cases, since in the whole calculation we have admitted no arbitrary limitation. The whole artifice consisted in rendering our equation divisible by $t^2 + 3u^2$; for we have thus been able to determine the letters t and u by an equation of the first degree: and innumerable applications may be made of these formulæ, some of which we shall give for the sake of example.

1. Let $k = 0$, and $h = 1$, we shall have $t = -3g(f^2 + 3g^2)$, and $u = f(f^2 + 3g^2) - 1$; so that $p = -3fg(f^2 + 3g^2) + 3fg(f^2 + 3g^2) - 3g$, or $p = -3g$; $q = -(f^2 + 3g^2)^2 + f$; $s = -3g(f^2 + 3g^2)$; $r = -f(f^2 + 3g^2) + 1$; consequently,

$$x = -3g - (f^2 + 3g^2)^2 + f,$$
$$y = -3g + (f^2 + 3g^2)^2 - f,$$
$$z = (3g - f) \times (f^2 + 3g^2) + 1;$$

lastly, $v = -(3g + f) \times (f^2 + 3g^2) + 1$.

If we also suppose $f = -1$, and $g = +1$, we shall have $x = -20$, $y = 14$, $z = 17$, and $v = -7$; and thence results the final equation, $-20^3 + 14^3 + 17^3 = -7^3$, or $14^3 + 17^3 + 7^3 = 20^3$.

2. Let $f = 2$, $g = 1$, and consequently $f^2 + 3g^2 = 7$; farther, $h = 0$, and $k = 1$; so that $h^2 + 3k^2 = 3$; we shall then have $t = -12$, and $u = 14$; so that

$$p = 2t + 3u = 18, \qquad q = t - 2u = -40,$$
$$r = t = -12, \qquad \text{and } s = 3u = 42.$$

From this will result

$$x = p + q = -22, \qquad y = p - q = 58,$$
$$z = r - s = -54, \qquad \text{and } v = r + s = 30;$$

therefore, $30^3 = 22^3 + 58^3 - 54^3$, or
$$58^3 = 30^3 + 54^3 + 22^3;$$

and as all these roots are divisible by 2, we shall also have

$$29^3 = 15^3 + 27^3 + 11^3.$$

3. Let $f = 3$, $g = 1$, $h = 1$, and $k = 1$; so that $f^2 + 3g^2 = 12$, $h^2 + 3k^2 = 4$; also $t = -24$, and $u = 32$. Here, these two values being divisible by 8, and as we consider only their ratios, we may make $t = -3$, and $u = 4$. Whence we obtain

$$p = 3t + 3u = +3, \qquad q = t - 3u = -15,$$
$$r = t - u = -7, \quad \text{and } s = t + 3u = +9;$$

consequently, $x = -12$, and $y = 18$,
$z = -16$, and $v = 2$,

whence $-12^3 + 18^3 - 16^3 = 2^3$, or $18^3 = 16^3 + 12^3 + 2^3$, or, dividing by the cube of 2, $9^3 = 8^3 + 6^3 + 1^3$.

4. Let us also suppose $g = 0$, and $k = h$, by which means we leave f and h undetermined. We shall thus have $f^2 + 3g^2 = f^2$, and $h^2 + 3k^2 = 4h^2$; so that $t = 12h^3$, and $u = f^3 - 4h^3$; also, $p = st = 12fh^3$, $q = -f^4 + 4fh^3$, $r = 12h^4 - hf^3 + 4h^4 = 16h^4 - hf^3$, and $s = 3hf^3$; lastly,

$$x = p + q = 16fh^3 - f^4, \qquad y = p - q = 8fh^3 + f^4,$$
$$z = r - s = 16h^4 - 4hf^3, \text{ and } v = r + s = 16h^4 + 2hf^3.$$

If we now make $f = h = 1$, we have $x = 15$, $y = 9$, $z = 12$, and $v = 18$; or, dividing all by 3, $x = 5$, $y = 3$, $z = 4$, and $v = 6$; so that $3^3 + 4^3 + 5^3 = 6^3$. The progression of these three roots, 3, 4, 5, increasing by unity, is worthy of attention; for which reason, we shall investigate whether there are not others of the same kind.

249. *Question* 4. Required three numbers, whose difference is 1, and forming such an arithmetical progression, that their cubes added together may make a cube.

Let x be the middle number, or term, then $x - 1$ will be the least, and $x + 1$ the greatest; the sum of the cubes of these three numbers is $3x^3 + 6x = 3x(x^2 + 2)$, which must be a cube. Here, we must previously have a case, in which this property exists, and we find, after some trials, that that case is $x = 4$.

So that, according to the rules already given, we may make $x = 4 + y$; whence $x^2 = 16 + 8y + y^2$, and $x^3 = 64 + 48y + 12y^2 + y^3$, and by these means our formula becomes $216 + 150y + 36y^2 + 3y^3$, in which the first term is a cube, but the last is not.

Let us, therefore, suppose the root to be $6 + fy$, or the formula to be $216 + 108fy + 18f^2y^2 + f^3y^3$, and destroy the two second terms, by writing $108f = 150$, or $f = \frac{25}{18}$; the other terms, divided by y^2, will give

$$36 + 3y = 18f^2 + f^3y = \frac{25^2}{18} + \frac{25^3}{18^3}y, \text{ or}$$

$$18^3 \times 36 + 18^3 \times 3y = 18^2 \times 25^2 + 25^3y, \text{ or}$$
$$18^3 \times 36 - 18^2 \times 25^2 = 25^3y - 18^3 \times 3y; \text{ therefore}$$
$$y = \frac{18^3 \times 36 - 18^2 \times 25^2}{25^3 - 3 \times 18^3} = \frac{18^2 \times (18 \times 36 - 25^2)}{25^3 - 3 \times 18^3}; \text{ that}$$
is, $y = \frac{-324 \times 23}{1871} = \frac{-7452}{1871}$; and, consequently, $x = \frac{32}{1871}$.

As it might be difficult to pursue this reduction in cubes, it is proper to observe, that the question may always be reduced to squares. In fact, since $3x(x^2 + 2)$ must be a cube, let us suppose $3x(x^2 + 2) = x^3y^3$; dividing by x, we shall have $3x^2 + 6 = x^2y^3$; and, consequently,

$x^2 = \frac{6}{y^3 - 3} = \frac{36}{6y^3 - 18}$. Now, the numerator of this fraction being already a square, it is only necessary to transform the denominator, $6y^3 - 18$, into a square, which also requires that we have already found a case. For this purpose, let us consider that 18 is divisible by 9, but 6 only by 3, and that y therefore may be divided by 3; if we make $y = 3z$, our denominator will become $162z^3 - 18$, which being divided by 9, and becoming $18z^3 - 2$, must still be a square. Now, this is evidently true of the case $z = 1$. So that we shall make $z = 1 + v$, and we must have $16 + 54v + 54v^2 + 18v^3 = \square$. Let its root be $4 + \frac{27}{4}v$, the square of which is $16 + 54v + \frac{729}{16}v^2$, and we must have $54 + 18v = \frac{729}{16}$; or $18v = -\frac{135}{16}$, or $2v = -\frac{15}{16}$; and, consequently, $v = -\frac{15}{32}$; which produces $z = 1 + v = \frac{17}{32}$, and then $y = \frac{51}{32}$.

Let us now resume the denominator

$$6y^3 - 18 = 162z^3 - 18 = 9(18z^3 - 2);$$

and since the square root of the factor, $18z^3 - 2$, is $4 + \frac{27}{4}v = \frac{107}{128}$, that of the whole denominator is $\frac{321}{128}$: but the root of the numerator is 6; therefore $x = \frac{6}{\frac{321}{128}} = \frac{256}{107}$, a value quite different from that which we found before. It follows, therefore, that the roots of our three cubes sought are $x - 1 = \frac{149}{107}$, $x = \frac{256}{107}$, $x + 1 = \frac{363}{107}$: and the sum of the cubes of these three numbers will be a cube, whose root, $xy, = \frac{256}{107} \times \frac{51}{32} = \frac{13056}{3424} = \frac{408}{107}$.

250. We shall here finish this Treatise on the Indeterminate Analysis, having had sufficient occasion, in the questions which we have resolved, to explain the chief artifices that have hitherto been devised in this branch of Algebra.

QUESTIONS FOR PRACTICE.

1. To divide a square number (16) into two squares.

Ans. $\frac{256}{25}$, and $\frac{144}{25}$.

2. To find two square numbers, whose difference (60) is given. *Ans.* $72\frac{1}{4}$, and $132\frac{1}{4}$.

3. From a number x to take two given numbers 6 and 7, so that both remainders may be square numbers.

Ans. $x = \frac{121}{16}$.

4. To find two numbers in proportion as 8 is to 15, and such, that the sum of their squares shall make a square number. *Ans.* 576, and 1080.

5. To find four numbers such, that if the square number 100 be added to the product of every two of them, the sum shall be all squares. *Ans.* 12, 32, 88, and 168.

6. To find two numbers, whose difference shall be equal to the difference of their squares, and the sum of their squares a square number. *Ans.* $\frac{4}{7}$, and $\frac{3}{7}$.

7. To find two numbers, whose product being added to the sum of their squares, shall make a square number.

Ans. 5 and 3, 8 and 7, 16 and 5, &c.

8. To find two such numbers, that not only each number, but also their sum and their difference, being increased by unity, shall be square numbers. *Ans.* 3024, and 5624.

9. To find three square numbers such, that the sum of their squares shall be a square number.

Ans. 9, 16, and $\frac{144}{25}$.

10. To divide the cube number 8 into three other cube numbers. *Ans.* $\frac{64}{27}$, $\frac{125}{27}$, and 1.

11. Two cube numbers, 8 and 1, being given, to find two other cube numbers, whose difference shall be equal to the sum of the given cubes. *Ans.* $\frac{8000}{343}$, and $\frac{4913}{343}$.

12. To find three such cube numbers, that if 1 be subtracted from every one of them, the sum of the remainders shall be a square. *Ans.* $\frac{4913}{3375}$, $\frac{21952}{3375}$, and 8.

13. To find two numbers, whose sum shall be equal to the sum of their cubes. *Ans.* $\frac{5}{7}$, and $\frac{8}{7}$.

14. To find three such cube numbers, that the sum of them may be both a square and a cube.

Ans. 1, $\frac{2084383}{274625}$, $\frac{15252992}{274625}$.