such as $x^2 + y^2$, may be also a fifth power, we shall have $a = 1$, and $c = 1$; therefore, $x = p^5 - 10p^3q^2 + 5pq^4$; and $y = 5p^4q - 10p^2q^3 + q^5$; and, farther, making $p = 2$, and $q = 1$, we shall find $x = 38$, and $q = 41$; consequently,

$$x^2 + y^2 = 3125 = 5^5.$$

CHAP. XIII.

Of some Expressions *of the* Form $ax^4 + by^4$, *which are not reducible to* Squares.

202. Much labor has been formerly employed by some mathematicians to find two biquadrates, whose sum or difference might be a square, but in vain; and at length it has been demonstrated, that neither the formula $x^4 + y^4$, nor the formula $x^4 - y^4$, can become a square, except in these evident cases; first, when $x = 0$, or $y = 0$, and, secondly, when $y = x$. This circumstance is the more remarkable, because it has been seen, that we can find an infinite number of answers, when the question involves only simple squares.

203. We shall give the demonstration to which we have just alluded; and, in order to proceed regularly, we shall previously observe, that the two numbers x and y may be considered as prime to each other: for, if these numbers had a common divisor, so that we could make $x = dp$, and $y = dq$, our formulæ would become $d^4p^4 + d^4q^4$, and $d^4p^4 - d^4q^4$: which formulæ, if they were squares, would remain squares after being divided by d^4; therefore, the formulæ $p^4 + q^4$, and $p^4 - q^4$, also, in which p and q have no longer any common divisor, would be squares; consequently, it will be sufficient to prove, that our formulæ cannot become squares in the case of x and y being prime to each other, and our demonstration will, consequently, extend to all the cases, in which x and y have common divisors.

204. We shall begin, therefore, with the sum of two biquadrates; that is, with the formula $x^4 + y^4$, considering x and y as numbers that are prime to each other: and we have to prove, that this formula becomes a square only in the cases above-mentioned; in order to which, we shall enter

upon the analysis and deductions which this demonstration requires.

If any one denied the proposition, it would be maintaining that there may be such values of x and y, as will make x^4+y^4 a square, in great numbers, notwithstanding there are none in small numbers.

But it will be seen, that if x and y had satisfactory values, we should be able, however great those values might be, to deduce from them less values equally satisfactory, and from these, others still less, and so on. Since, therefore, we are acquainted with no value in small numbers, except the two cases already mentioned, which do not carry us any farther, we may conclude, with certainty, from the following demonstration, that there are no such values of x and y as we require, not even among the greatest numbers. The proposition shall afterwards be demonstrated, with respect to the difference of two biquadrates, x^4-y^4, on the same principle.

205. The following consideration, however, must be attended to at present, in order to be convinced that x^4+y^4 can only become a square in the self evident cases which have been mentioned.

1. Since we suppose x and y prime to each other, that is, having no common divisor, they must either both be odd, or one must be even, and the other odd.

2. But they cannot both be odd, because the sum of two odd squares can never be a square; for an odd square is always contained in the formula $4n+1$; and, consequently, the sum of two odd squares will have the form $4n+2$, which being divisible by 2, but not by 4, cannot be a square. Now, this must be understood also of two odd biquadrate numbers.

3. If, therefore, x^4+y^4 must be a square, one of the terms must be even and the other odd; and we have already seen, that, in order to have the sum of two squares a square, the root of one must be expressible by p^2-q^2, and that of the other by $2pq$; therefore, $x^2=p^2-q^2$, and $y^2=2pq$; and we should have $x^4+y^4=(p^2+q^2)^2$.

4. Consequently, y would be even, and x odd; but since $x^2=p^2-q^2$, the numbers p and q must also be the one even, and the other odd. Now, the first, p, cannot be even; for if it were, p^2-q^2 would be a number of the form $4n-1$, or $4n+3$, and could not become a square: therefore p must be odd, and q even, in which case it is evident, that these numbers will be prime to each other.

5. In order that p^2-q^2 may become a square, or

$p^2 - q^2 = x^2$, we must have, as we have already seen, $p = r^2 + s^2$, and $q = 2rs$; for then $x^2 = (r^2 - s^2)^2$, and $x = r^2 - s^2$.

6. Now, y^2 must likewise be a square; and since we had $y^2 = 2pq$, we shall now have $y^2 = 4rs(r^2 + s^2)$; so that this formula must be a square; therefore $rs(r^2 + s^2)$ must also be a square: and let it be observed, that r and s are numbers prime to each other; so that the three factors of this formula, namely, r, s, and $r^2 + s^2$, have no common divisor.

7. Again, when a product of several factors, that have no common divisor, must be a square, each factor must itself be a square; so that making $r = t^2$, and $s = u^2$, we must have $t^4 + u^4 = \square$.

If, therefore, $x^4 + y^4$ were a $\square$, our formula $t^4 + u^4$, which is, in like manner, the sum of two biquadrates, would also be a $\square$. And it is proper to observe here, that since $x^2 = t^4 - u^4$, and $y^2 = 4t^2u^2(t^4 + u^4)$ the numbers t and u will evidently be much smaller than x and y, since x and y are even determined by the fourth powers of t and u, and must therefore become much greater than these numbers.

8. It follows, therefore, that if we could assign, in numbers however great, two biquadrates, such as x^4 and y^4, whose sum might be a square, we could deduce from it a number, formed by the sum of two much less biquadrates, which would also be a square; and this new sum would enable us to find another of the same nature, still less, and so on, till we arrived at very small numbers. Now, such a sum not being possible in very small numbers, it evidently follows, that there is not one which we can express by very great numbers.

9. It might indeed be objected, that such a sum does exist in very small numbers; namely, in the case which we have mentioned, when one of the two biquadrates becomes nothing: but we answer, that we shall never arrive at this case, by coming back from very great numbers to the least, according to the method which has been explained; for if in the small sum, or the reduced sum, $t^4 - u^4$, we had $t = 0$, or $u = 0$, we should necessarily have $y^2 = 0$ in the great sum; but this is a case which does not here enter into consideration.

206. Let us proceed to the second proposition, and prove also that the difference of two biquadrates, or $x^4 - y^4$, can never become a square, except in the cases of $y = 0$, and $y = x$.

1. We may consider the numbers x and y as prime to each other, and consequently, as being either both odd, or

the one even and the other odd: and as in both cases the difference of two squares may become a square, we must consider these two cases separately.

2. Let us, therefore, first begin by supposing both the numbers x and y odd, and that $x = p + q$, and $y = p - q$; then one of the two numbers p and q must necessarily be even, and the other odd. We have also $x^2 - y^2 = 4pq$, and $x^2 + y^2 = 2p^2 + 2q^2$; therefore our formula $x^4 - y^4 = 4pq(2p^2 + 2q^2)$; and as this must be a square, its fourth part, $pq(2p^2 + 2q^2) = 2pq(p^2 + q^2)$, must also be a square. Also, since the factors of this formula have no common divisor (because if p is even, q must be odd), each of these factors, $2p$, q, and $p^2 + q^2$, must be a square. In order, therefore, that the first two may become squares, let us suppose $2p = 4r^2$, or $p = 2r^2$, and $q = s^2$; in which s must be odd, and the third factor, $4r^4 + s^4$, must likewise be a square.

3. Now, since $s^4 + 4r^4$ is the sum of two squares, the first of which, s^4, is odd, and the other, $4r^4$, is even, let us make the root of the first $s^2 = t^2 - u^2$, in which let t be odd, and u even; and the root of the second, $2r^2 = 2tu$, or $r^2 = tu$, where t and u are prime to each other.

4. Since $tu = r^2$ must be a square, both t and u must be squares also. If, therefore, we suppose $t = m^2$, and $u = n^2$, (representing an odd number by m, and an even number by n), we shall have $s^2 = m^4 - n^4$; so that here, also, it is required to make the difference of two biquadrates, namely, $m^4 - n^4$, a square. Now, it is obvious, that these numbers would be much less than x and y, since they are less than r and s, which are themselves evidently less than x and y. If a solution, therefore, were possible in great numbers, and $x^4 - y^4$ were a square, there must also be one possible for numbers much less; and this last would lead us to another solution for numbers still less, and so on.

5. Now, the least numbers for which such a square can be found, are in the case where one of the biquadrates is 0, or where it is equal to the other biquadrate. In the first case, we must have $n = 0$; therefore $u = 0$, and also $r = 0$, $p = 0$, and, lastly, $x^4 - y^4 = 0$, or $x^4 = y^4$; which is a case that does not belong to the present question; if $n = m$, we shall find $t = u$, then $s = 0$, $q = 0$, and, lastly, also $x = y$, which does not here enter into consideration.

207. It might be objected, that since m is odd, and n even, the last difference is no longer similar to the first; and that, therefore, we can form no analogous conclusions from it with respect to smaller numbers. But it is sufficient that the first difference has led us to the second; and we shall

shew, that $x^4 - y^4$ can no longer become a square, when one of the biquadrates is even, and the other odd.

1. We may observe, if the first term, x^4, were even, and y^4 odd, the impossibility of the thing would be self-evident, since we should have a number of the form $4n + 3$; which cannot be a square: therefore, let x be odd, and y even; then $x^2 = p^2 + q^2$, and $y = 2pq$; whence $x^4 - y^4 = p^4 - 2p^2q^2 + q^4 = (p^2 - q^2)^2$, where one of the two numbers p and q must be even, and the other odd.

2. Now, as $p^2 + q^2 = x^2$ must be a square, we have $p = r^2 - s^2$, and $q = 2rs$; whence $x = r^2 + s^2$: but from that results $y^2 = 2(r^2 - s^2) \times 2rs$, or $y^2 = 4rs \times (r^2 - s^2)$, and as this must be a square, its fourth part, $rs(r^2 - s^2)$, whose factors are prime to each other, must likewise be a square.

3. Let us, therefore, make $r = t^2$, and $s = u^2$, and we shall have the third factor $r^2 - s^2 = t^4 - u^4$, which must also be a square. Now, as this factor is equal to the difference of two biquadrates, which are much less than the first, the preceding demonstration is fully confirmed; and it is evident, that, if the difference of two biquadrates could become equal to the square of a number (however great we may suppose it), we could, by means of this known case, arrive at differences less and less, which would also be reducible to squares, without our being led back to the two evident cases mentioned at first. It is impossible, therefore, for the thing to take place even with respect to the greatest numbers.

208. The first part of the preceding demonstration, namely, where x and y are supposed odd, may be abridged as follows: if $x^4 - y^4$ were a square, we must have $x^2 = p^2 + q^2$, and $y^2 = p^2 - q^2$, representing by p and q numbers, the one of which is even and the other odd; and by these means we should obtain $x^2y^2 = p^4 - q^4$; and, consequently, $p^4 - q^4$ must be a square. Now, this is a difference of two biquadrates, the one of which is even and the other odd; and it has been proved, in the second part of the demonstration, that such a difference cannot become a square.

209. We have therefore proved these two principal propositions; that neither the sum, nor the difference, of two biquadrates, can become a square number, except in a very few self-evident cases.

Whatever formulæ, therefore, we wish to transform into squares, if those formula require us to reduce the sum, or the difference of two biquadrates to a square, it may be pronounced that the given formulæ are likewise impossible;

which happens with regard to those that we shall now point out.

1. It is not possible for the formula $x^4 + 4y^4$ to become a square; for since this formula is the sum of two squares, we must have $x^2 = p^2 - q^2$, and $2y^2 = 2pq$, or $y^2 = pq$; now p and q being numbers prime to each other, each of them must be a $\square$. If we therefore make $p = r^2$, and $q = s^2$, we shall have $x^2 = r^4 - s^4$; that is to say, the difference of two biquadrates must be a square, which is impossible.

2. Nor is it possible for the formula $x^4 - 4y^4$ to become a square; for in this case we must make $x^2 = p^2 + q^2$, and $2y^2 = 2pq$, that we may have $x^4 - 4y^4 = (p^2 - q^2)^2$; but, in order that $y^2 = pq$, both p and q must be squares: and if we therefore make $p = r^2$, and $q = s^2$, we have $x^2 = r^4 + s^4$; that is to say, the sum of two biquadrates must be reducible to a square, which is impossible.

3. It is impossible also for the formula $4x^4 - y^4$ to become a square, because in this case y must necessarily be an even number. Now, if we make $y = 2z$, we conclude that $4x^4 - 16z^4$, and consequently, also, its fourth part, $x^4 - 4z^4$, must be reducible to a square; which we have just seen is impossible.

4. The formula $2x^4 + 2y^4$ cannot be transformed into a square; for since that square would necessarily be even, and consequently, $2x^4 + 2y^4 = 4z^2$, we should have $x^4 + y^4 = 2z^2$, or $2z^2 + 2x^2y^2 = x^4 + 2x^2y^2 + y^4 = \square$; or, in like manner, $2z^2 - 2x^2y^2 = x^4 - 2x^2y^2 + y^4 = \square$. So that, as both $2z^2 + 2x^2y^2$, and $2z^2 - 2x^2y^2$, would become squares, their product, $4z^4 - 4x^4y^4$, as well as the fourth of that product, or $z^4 - x^4y^4$, must be a square. But this last is the difference of two biquadratics; and is therefore impossible.

5. Lastly, I say also that the formula $2x^4 - 2y^4$ cannot be a square; for the two numbers x and y cannot both be even, since, if they were, they would have a common divisor; nor can they be the one even and the other odd, because then one part of the formula would be divisible by 4, and the other only by 2; and thus the whole formula would only be divisible by 2; therefore these numbers x and y must both be odd. Now, if we make $x = p + q$, and $y = p - q$, one of the numbers p and q will be even and the other will be odd; and, since $2x^4 - 2y^4 = 2(x^2 + y^2) \times (x^2 - y^2)$, and $x^2 + y^2 = 2p^2 + 2q^2 = 2(p^2 + q^2)$, and $x^2 - y^2 = 4pq$, our formula will be expressed by $16pq(p^2 + q^2)$, the sixteenth part of which, or $pq(p^2 + q^2)$, must likewise be a square. But these factors are prime to each other, so that each of

them must be a square. Let us, therefore, make the first two $p = r^2$, and $q = s^2$, and the third will become $r^4 + s^4$, which cannot be a square, therefore the given formula cannot become a square.

210. We may likewise demonstrate, that the formula $x^4 + 2y^4$ can never become a square: the *rationale* of this demonstration being as follows:

1. The number x cannot be even, because in that case y must be odd; and the formula would only be divisible by 2, and not by 4; so that x must be odd.

2. If, therefore, we suppose the square root of our formula to be $x^2 + \frac{2py^2}{q}$, in order that it may become odd, we shall have $x^4 + 2y^4 = x^4 + \frac{4px^2y^2}{q} + \frac{4p^2y^4}{q^2}$, in which the terms x^4 are destroyed; so that if we divide the other terms by y^2, and multiply by q^2, we find $4pqx^2 + 4p^2y^2 = 2q^2y^2$, or $4pqx^2 = 2q^2y^2 - 4p^2y^2$, whence we obtain $\frac{x^2}{y^2} = \frac{q^2 - 2p^2}{2pq}$; that is, $x^2 = q^2 - 2p^2$, and $y^2 = 2pq$*, which are the same formulæ that have been already given.

3. So that $q^2 - 2p^2$ must be a square, which cannot happen, unless we make $q = r^2 + 2s^2$, and $p = 2rs$, in order to have $x^2 = (r^2 - 2s^2)^2$; now, this will give us $4rs(r^2 + 2s^2) = y^2$; and its fourth part, $rs(r^2 + 2s^2)$ must also be a square: consequently r and s must respectively be each a square. If, therefore, we suppose $r = t^2$, and $s = u^2$, we shall find the third factor $r^2 + 2s^2 = t^4 + 2u^4$, which ought to be a square.

4. Consequently, if $x^4 + 2y^4$ were a square, $t^4 + 2u^4$ must also be a square; and as the numbers t and u would be much less than x and y, we should always come, in the same manner, to numbers successively less: but as it is easy from trials to be convinced, that the given formula is not a square in any small number; it cannot therefore be the square of a very great number.

211. On the contrary, with regard to the formula $x^4 - 2y^4$, it is impossible to prove that it cannot become a square; and, by a process of reasoning similar to the foregoing, we even find that there are an infinite number of cases in which this formula really becomes a square.

In fact, if $x^4 - 2y^4$ must become a square, we shall see

* Because x and y are prime to each other.

that, by making $x^2 = p^2 + 2q^2$, and $y^2 = 2pq$, we find $x^4 - 2y^4 = (p^2 - 2q^2)^2$. Now, $p^2 + 2q^2$ must in that case evidently become a square; and this happens when $p = r^2 - 2s^2$, and $q = 2rs$; since we have, in this case, $x^2 = (r^2 + 2s^2)^2$; and farther, it is to be observed, that, for the same purpose, we may take $p = 2s^2 - r^2$, and $q = 2rs$. We shall therefore consider each case separately.

1. First, let $p = r^2 - 2s^2$, and $q = 2rs$; we shall then have $x = r^2 + 2s^2$; and, since $y^2 = 2pq$, we shall thus have $y^2 = 4rs(r^2 - 2s^2)$; so that r and s must be squares: making, therefore, $r = t^2$, and $s = u^2$, we shall find $y^2 = 4t^2u^2(t^4 - 2u^4)$. So that $y = 2tu\sqrt{(t^4 - 2u^4)}$, and $x = t^4 + 2u^4$; therefore, when $t^4 - 2u^4$ is a square, we shall also find $x^4 - 2y^4 = \square$; but although t and u are numbers less than x and y, we cannot conclude that it is impossible for $x^4 - 2y^4$ to become a square, from our arriving at a similar formula in smaller numbers; since $x^4 - 2y^4$ may become a square, without our being brought to the formula $t^4 - 2y^4$, as will be seen by considering the second case.

2. For this purpose, let $p = 2s^2 - r^2$, and $q = 2rs$. Here, indeed, as before, we shall have $x = r^2 + 2s^2$; but then we shall find $y^2 = 2pq = 4rs(2s^2 - r^2)$: and if we suppose $r = t^2$, and $s = u^2$, we obtain $y^2 = 4t^2u^2(2u^4 - t^4)$; consequently, $y = 2tu\sqrt{(2u^4 - t^4)}$, and $x = t^4 + 2u^4$, by which means it is evident that our formula $x^4 - 2y^4$ may also become a square, when the formula $2u^4 - t^4$ becomes a square. Now, this is evidently the case, when $t = 1$, and $u = 1$; and we from that obtain $x = 3$, $y = 2$, and, lastly,

$$x^4 - 2y^4 = 81 - (2 \times 16) = 49.$$

3. We have also seen, Art. 140, that $2u^4 - t^4$ becomes a square, when $u = 13$, and $t = 1$; since then $\sqrt{(2u^4 - t^4)} = 239$. If we substitute these values instead of t and u, we find a new case for our formula; namely, $x = 1 + 2 \times 13^4 = 57123$, and $y = 2 \times 13 \times 239 = 6214$.

4. Farther, since we have found values of x and y, we may substitute them for t and u in the foregoing formulæ, and shall obtain by these means new values of x and y.

Now, we have just found $x = 3$, and $y = 2$; let us, therefore, in the formulæ, (No. 1.) make $t = 3$, and $u = 2$; so that $\sqrt{(t^4 - 2u^4)} = 7$, and we shall have the following new values; $x = 81 + (2 \times 16) = 113$, and $y = 2 \times 3 \times 2 \times 7 = 84$; so that $x^2 = 12769$, and $x^4 = 163047361$. Farther, $y^2 = 7056$, and $y^4 = 49787136$; therefore $x^4 - 2y^4 = 63473089$: the square root of which number is 7967, and it agrees perfectly with the formula which was adopted at first, $p^2 - 2q^2$; for since $t = 3$, and $u = 2$, we have $r = 9$, and $s = 4$; wherefore $p = 81 - 32 = 49$, and $q = 72$; whence $p^2 - 2q^2 = 2401 - 10368 = -7967$.

CHAP. XIV.

Solution *of some* Questions *that belong to this part of* Algebra.

212. We have hitherto explained such artifices as occur in this part of Algebra, and such as are necessary for resolving any question belonging to it: it remains to make them still more clear, by adding here some of those questions with their solutions.

213. *Question* 1. To find such a number, that if we add unity to it, or subtract unity from it, we may obtain in both cases a square number.

Let the number sought be x; then both $x + 1$, and $x - 1$ must be squares. Let us suppose for the first case $x+1=p^2$, we shall have $x = p^2 - 1$, and $x - 1 = p^2 - 2$, which must likewise be a square. Let its root, therefore, be represented by $p - q$; and we shall have $p^2 - 2 = p^2 - 2pq + q^2$; consequently, $p = \frac{q^2 + 2}{2q}$. Hence we obtain $x = \frac{q^4 + 4}{4q^2}$, in which we may give q any value whatever, even a fractional one.

If we therefore make $q = \frac{r}{s}$, so that $x = \frac{r^4 + 4s^4}{4r^2s^2}$, we shall have the following values for some small numbers:

If $r = 1$,	2,	1,	3,	4,
and $s = 1$,	1,	2,	1,	1,
we have $x = \frac{5}{4}$,	$\frac{5}{4}$,	$\frac{65}{16}$,	$\frac{85}{36}$,	$\frac{65}{16}$.

214. *Question* 2. To find such a number x, that if we add to it any two numbers, for example, 4 and 7, we obtain in both cases a square.

According to this enunciation, the two formulæ, $x + 4$ and $x + 7$, must become squares. Let us therefore suppose the first $x + 4 = p^2$, which gives us $x = p^2 - 4$, and the