ELEMENTS

OF

ALGEBRA.

PART II.

Containing the Analysis *of* Indeterminate Quantities.

CHAP. I.

Of the Resolution *of* Equations *of the* First Degree, *which contain more than one unknown* Quantity.

ARTICLE I.

IT has been shewn, in the First Part, how one unknown quantity is determined by a single equation, and how we may determine two unknown quantities by means of two equations, three unknown quantities by three equations, and so on; so that there must always be as many equations as there are unknown quantities to determine, at least when the question itself is determinate.

When a question, therefore, does not furnish as many equations as there are unknown quantities to be determined, some of these must remain undetermined, and depend on our will; for which reason, such questions are said to be *indeterminate*; forming the subject of a particular branch of algebra, which is called *Indeterminate Analysis*.

2. As in those cases we may assume any numbers for one, or more unknown quantities, they also admit of several solutions: but, on the other hand, as there is usually annexed the condition, that the numbers sought are to be integer and positive, or at least rational, the number of all the possible solutions of those questions is greatly limited: so that often there are very few of them possible; at other

times, there may be an infinite number, but such as are not readily obtained; and sometimes, also, none of them are possible. Hence it happens, that this part of analysis frequently requires artifices entirely appropriate to it, which are of great service in exercising the judgment of beginners, and giving them dexterity in calculation.

3. To begin with one of the easiest questions. Let it be required to find two positive, integer numbers, the sum of which shall be equal to 10.

Let us represent those members by x and y; then we have $x + y = 10$; and $x = 10 - y$, where y is so far only determined, that this letter must represent an integer and positive number. We may therefore substitute for it all integer numbers from 1 to infinity: but since x must likewise be a positive number, it follows, that y cannot be taken greater than 10, for otherwise x would become negative; and if we also reject the value of $x = 0$, we cannot make y greater than 9; so that only the following solutions can take place:

$$\text{If } y = 1, 2, 3, 4, 5, 6, 7, 8, 9,$$
$$\text{then } x = 9, 8, 7, 6, 5, 4, 3, 2, 1.$$

But, the last four of these nine solutions being the same as the first four, it is evident, that the question really admits only of five different solutions.

If three numbers were required, the sum of which might make 10, we should have only to divide one of the numbers already found into two parts, by which means we should obtain a greater number of solutions.

4. As we have found no difficulty in this question, we will proceed to others, which require different considerations.

Question 1. Let it be required to divide 25 into two parts, the one of which may be divisible by 2, and the other by 3.

Let one of the parts sought be $2x$, and the other $3y$; we shall then have $2x + 3y = 25$; consequently $2x = 25 - 3y$; and dividing by 2, we obtain

$x = \frac{25 - 3y}{2}$; whence we conclude, in the first place, that $3y$ must be less than 25, and, consequently, y is less than 8. Also, if, from this value of x, we take out as many integers as we possibly can, that is to say, if we divide by the denominator 2, we shall have $x = 12 - y + \frac{1 - y}{2}$; whence it follows, that $1 - y$, or rather $y - 1$, must be divisible by by 2. Let us, therefore, make $y - 1 = 2z$; and we shall have $y = 2z + 1$, so that

$$x = 12 - 2z - 1 - z = 11 - 3z.$$

And, since y cannot be greater than 8, we must not substitute any numbers for z which would render $2z + 1$ greater than 8; consequently, z must be less than 4, that is to say, z cannot be taken greater than 3, for which reasons we have the following answers:

If we make	$z = 0$	$z = 1$	$z = 2$	$z = 3$,
we have	$y = 1$	$y = 3$	$y = 5$	$y = 7$,
and	$x = 11$	$x = 8$	$x = 5$	$x = 2$.

Hence, the two parts of 25 sought, are

$$22 + 3,\ 16 + 9,\ 10 + 15, \text{ or } 4 + 21.$$

5. *Question* 2. To divide 100 into two such parts, that the one may be divisible by 7, and the other by 11.

Let $7x$ be the first part, and $11y$ the second. Then we must have $7x + 11y = 100$; and, consequently,

$$x = \frac{100 - 11y}{7} = \frac{98 + 2 - 7y - 4y}{7}, \text{ or}$$

$$x = 14 - y + \frac{2 - 4y}{7};$$

wherefore $2 - 4y$, or $4y - 2$, must be divisible by 7.

Now, if we can divide $4y - 2$ by 7, we may also divide its half, $2y - 1$, by 7. Let us therefore make $2y - 1 = 7z$, or $2y = 7z + 1$, and we shall have $x = 14 - y - 2z$; but, since $2y = 7z + 1 = 6z + z + 1$, we shall have $y = 3z + \frac{z+1}{2}$. Let us therefore make $z + 1 = 2u$, or $z = 2u - 1$; which supposition gives $y = 3z + u$; and, consequently, we may substitute for u every integer number that does not make x or y negative. Now, as y becomes $= 7u - 3$, and $x = 19 - 11u$, the first of these expressions shews that $7u$ must exceed 3; and according to the second, $11u$ must be less than 19, or u less than $\frac{19}{11}$: so that u cannot be 2; and since it is impossible for this number to be 0, we must have $u = 1$: which is the only value that this letter can have. Hence, we obtain $x = 8$, and $y = 4$; and the two parts of 100 which were required, are 56, and 44.

6. *Question* 3. To divide 100 into two such parts, that dividing the first by 5, there may remain 2; and dividing the second by 7, the remainder may be 4.

Since the first part, divided by 5, leaves the remainder 2, let us suppose it to be $5x + 2$; and, for a similar reason, we may represent the second part by $7y + 4$: we shall thus have

$$5x + 7y + 6 = 100, \text{ or } 5x = 94 - 7y = 90 + 4 - 5y - 2y;$$

whence we obtain $x = 18 - y + \frac{4-2y}{5}$. Hence it follows, that $4 - 2y$, or $2y - 4$, or the half $y - 2$, must be divisible by 5. For this reason, let us make $y - 2 = 5z$, or $y = 5z + 2$, and we shall have $x = 16 - 7z$; whence we conclude, that $7z$ must be less than 16, and z less than $\frac{16}{7}$, that is to say, z cannot exceed 2. The question proposed, therefore, admits of three answers:

1. $z = 0$ gives $x = 16$, and $y = 2$; whence the two parts are $82 + 18$.
2. $z = 1$ gives $x = 9$, and $y = 7$; and the two parts are $47 + 53$.
3. $z = 2$ gives $x = 2$, and $y = 12$; and the two parts are $12 + 88$.

7. *Question* 4. Two women have together 100 eggs: one says to the other; 'When I count my eggs by eights, there is an overplus of 7.' The second replies: 'If I count mine by tens, I find the same overplus of 7.' How many eggs had each?

As the number of eggs belonging to the first woman, divided by 8, leaves the remainder 7; and the number of eggs belonging to the second, divided by 10, gives the same remainder 7; we may express the first number by $8x + 7$, and the second by $10y + 7$; so that $8x + 10y + 14 = 100$, or $8x = 86 - 10y$, or $4x = 43 - 5y = 40 + 3 - 4y - y$. Consequently, if we make $y - 3 = 4z$, so that $y = 4z + 3$, we shall have

$$x = 10 - 4z - 3 - z = 7 - 5z;$$

whence it follows, that $5z$ must be less than 7, or z less than 2; that is to say, we shall only have the two following answers:

1. $z = 0$ gives $x = 7$, and $y = 3$; so that the first woman had 63 eggs, and the second 37.
2. $z = 1$ gives $x = 2$, and $y = 7$; therefore the first woman had 23 eggs, and the second had 77.

8. *Question* 5. A company of men and women spent 1000 sous at a tavern. The men paid each 19 sous, and each woman 13. How many men and women were there?

Let the number of men be x, and that of the women y, we shall then have the equation

$$19x + 13y = 1000, \text{ or}$$
$$13y = 1000 - 19x = 988 + 12 - 13x - 6x, \text{ and}$$
$$y = 76 - x + \frac{12-6x}{13};$$

whence it follows, that $12 - 6x$, or $6x - 12$, or $x - 2$, the sixth part of that number must be divisible by 13. If, therefore, we make $x - 2 = 13z$, we shall have $x = 13z + 2$,

$$\text{and } y = 76 - 13z - 2 - 6z, \text{ or } y = 74 - 19z;$$

which shews that z must be less than $\frac{74}{19}$, and, consequently, less than 4; so that the four following answers are possible:

1. $z = 0$ gives $x = 2$, and $y = 74$: in which case there were 2 men and 74 women; the former paid 38 sous, and the latter 962 sous.

2. $z = 1$ gives the number of men $x = 15$, and that of women $y = 55$; so that the former spent 285 sous, and the latter 715 sous.

3. $z = 2$ gives the number of men $x = 28$, and that of the women $y = 36$; therefore the former spent 532 sous, and the latter 468 sous.

4. $z = 3$ gives $x = 41$, and $y = 17$; so that the men spent 779 sous, and the women 221 sous.

9. *Question* 6. A farmer lays out the sum of 1770 crowns in purchasing horses and oxen; he pays 31 crowns for each horse, and 21 crowns for each ox. How many horses and oxen did he buy?

Let the number of horses be x, and that of oxen y; we shall then have $31x + 21y = 1770$, or $21y = 1770 - 31x = 1764 + 6 - 21x - 10x$; that is to say,

$y = 84 - x + \frac{6-10x}{21}$. Therefore $10x - 6$, and likewise its half $5x - 3$, must be divisible by 21. If we now suppose $5x - 3 = 21z$, we shall have $5x = 21z + 3$, and hence $y = 84 - x - 2z$. But, since

$x = \frac{21z+3}{5} = 4z + \frac{z+3}{5}$, we must also make $z + 3 = 5u$; which supposition gives

$$z = 5u - 3,\ x = 21u - 12, \text{ and}$$
$$y = 84 - 21u + 12 - 10u + 6 = 102 - 31u;$$

hence it follows, that u must be greater than 0, and yet less than 4, which furnishes the following answers:

1. $u = 1$ gives the number of horses $x = 9$, and that of oxen $y = 71$; wherefore the former cost 279 crowns, and the latter 1491; in all 1770 crowns.

2. $u = 2$ gives $x = 30$, and $y = 40$; so that the horses cost 930 crowns, and the oxen 840 crowns, which together make 1770 crowns.

3. $u = 3$ gives the number of the horses $x = 51$, and that of the oxen $y = 9$; the former cost 1581 crowns, and the latter 189 crowns; which together make 1770 crowns.

10. The questions which we have hitherto considered lead all to an equation of the form $ax + by = c$, in which a, b, and c, represent integer and positive numbers, and in which the values of x and y must likewise be integer and positive. Now, if b is negative, and the equation has the form $ax - by = c$, we have questions of quite a different kind, admitting of an infinite number of answers, which we shall treat of before we conclude the present chapter.

The simplest questions of this sort are such as the following. Required two numbers, whose difference may be 6. If, in this case, we make the less number x, and the greater y, we must have $y - x = 6$, and $y = 6 + x$. Now, nothing prevents us from substituting, instead of x, all the integer numbers possible, and whatever number we assume, y will always be greater by 6. Let us, for example, make $x = 100$, we have $y = 106$; it is evident, therefore, that an infinite number of answers are possible.

11. Next follow questions, in which $c = 0$, that is to say, in which ax must simply be equal to by. Let there be required, for example, a number divisible both by 5 and by 7. If we write N for that number, we shall first have $N = 5x$, since N must be divisible by 5; farther, we shall have $N = 7y$, because the number must also be divisible by 7; we shall therefore have $5x = 7y$, and $x = \frac{7y}{5}$. Now, since 7 cannot be divided by 5, y must be divisible by 5: let us therefore make $y = 5z$, and we have $x = 7z$; so that the number sought $N = 35z$; and as we may take for z, any integer number whatever, it is evident that we can assign for N an infinite number of values; such as

35, 70, 105, 140, 175, 210, &c.

If, beside the above condition, it were also required that the number N be divisible by 9, we should first have $N = 35z$, as before, and should farther make $N = 9u$. In this manner, $35z = 9u$, and $u = \frac{35z}{9}$; where it is evident that z must be divisible by 9; therefore let $z = 9s$; and we shall then have $u = 35s$, and the number sought $\text{N} = 315s$.

12. We find more difficulty, when c is not $= 0$. For example, when $5x = 7y + 3$, the equation to which we are led, and which requires us to seek a number N such, that it may be divisible by 5, and if divided by 7, may leave the remainder 3: for we must then have $\text{N} = 5x$, and also $\text{N} = 7y + 3$, whence results the equation $5x = 7y + 3$; and, consequently,

$$x = \frac{7y+3}{5} = \frac{5y+2y+3}{5} = y + \frac{2y+3}{5}.$$

If we make $2y + 3 = 5z$, or $z = \frac{2y+3}{5}$, we have $x = y + z$; now, because $2y + 3 = 5z$, or $2y = 5z - 3$, we have

$$y = \frac{5z-3}{2}, \text{ or } y = 2z + \frac{z-3}{2}.$$

If, therefore, we farther suppose $z - 3 = 2u$, we have $z = 2u + 3$, and $y = 5u + 6$, and

$$x = y + z = (5u + 6) + (2u + 3) = 7u + 9.$$

Hence, the number sought $\text{N} = 35u + 45$, in which equation we may substitute for u not only all positive integer numbers, but also negative numbers; for, as it is sufficient that N be positive, we may make $u = -1$, which gives $\text{N} = 10$; the other values are obtained by continually adding 35; that is to say, the numbers sought are 10, 45, 80, 115, 150, 185, 220, &c.

13. The solution of questions of this sort depends on the relation of the two numbers by which we are to divide; that is, they become more or less tedious, according to the nature of those divisors. The following question, for example, admits of a very short solution:

Required a number which, divided by 6, leaves the remainder 2; and divided by 13, leaves the remainder 3.

Let this number be N; first $\text{N} = 6x + 2$, and then $\text{N} = 13y + 3$; consequently, $6x + 2 = 13y + 3$, and $6x = 13y + 1$; hence,

$$x = \frac{13y+1}{6} = 2y + \frac{y+1}{6},$$

and if we make $y + 1 = 6z$, we obtain $y = 6z - 1$, and $x = 2y + z = 13z - 2$; whence we have for the number sought $N = 78z - 10$; therefore, the question admits of the following values of N; viz.

$$N = 68,\ 146,\ 224,\ 302,\ 380,\ \&c.$$

which numbers form an arithmetical progression, whose difference is $78 = 6 \times 13$. So that if we know one of the values, we may easily find all the rest; for we have only to add 78 continually, or to subtract that number, as long as it is possible, when we seek for smaller numbers.

14. The following question furnishes an example of a longer and more tedious solution.

Question 8. To find a number N, which, when divided by 39, leaves the remainder 16; and such also, that if it be divided by 56, the remainder may be 27.

In the first place, we have $N = 39p + 16$; and in the second, $N = 56q + 27$; so that

$$39p + 16 = 56q + 27, \text{ or } 39p = 56q + 11, \text{ and}$$

$$p = \frac{56q+11}{39} = q + \frac{17q+11}{39} = q + r, \text{ by making}$$

$$r = \frac{17q+11}{39}. \quad \text{So that } 39r = 17q + 11, \text{ and}$$

$$q = \frac{39r-11}{17} = 2r + \frac{5r-11}{17} = 2r + s, \text{ by making}$$

$$s = \frac{5r-11}{17}, \text{ or } 17s = 5r - 11; \text{ whence we get}$$

$$r = \frac{17s+11}{5} = 3s + \frac{2s+11}{5} = 3s + t, \text{ by making}$$

$$t = \frac{2s+11}{5}, \text{ or } 5t = 2s + 11; \text{ whence we find}$$

$$s = \frac{5t-11}{2} = 2t + \frac{t-11}{2} = 2t + u, \text{ by making}$$

$$u = \frac{t-11}{2}; \text{ whence } t = 2u + 11.$$

Having now no longer any fractions, we may take u at pleasure, and then we have only to trace back the following values:

$$\begin{aligned} t &= 2u + 11, \\ s = 2t + u &= 5u + 22, \\ r = 3s + t &= 17u + 77, \\ q = 2r + s &= 39u + 176, \\ p = q + r &= 56u + 253, \end{aligned}$$

and, lastly, N $= 39 \times 56u + 9883$ *. And the least possible value of N is found by making $u = -4$; so by this supposition, we have N $= 1147$: and if we make $u = x - 4$, we find

N $= 2184x - 8736 + 9883$; or N $= 2184x + 1147$;

which numbers form an arithmetical progression, whose first term is 1147, and whose common difference is 2184; the following being some of its leading terms:

1147, 3331, 5515, 7699, 9883, &c.

15. We shall subjoin some other questions by way of practice.

Question 9. A company of men and women club together for the payment of a reckoning: each man pays 25 livres, and each woman 16 livres; and it is found that all the women together have paid 1 livre more than the men. How many men and women were there?

Let the number of women be p, and that of men q; then the women will have expended $16p$, and the men $25q$; so that $16p = 25q + 1$, and

$$p = \frac{25q+1}{16} = q + \frac{9q+1}{16} = q + r, \text{ or } 16r = 9q + 1,$$

$$q = \frac{16r-1}{9} = r + \frac{7r-1}{9} = r + s, \text{ or } 9s = 7r - 1,$$

$$r = \frac{9s+1}{7} = s + \frac{2s+1}{7} = s + t, \text{ or } 7t = 2s + 1,$$

$$s = \frac{7t-1}{2} = 3s + \frac{t-1}{2} = 3s + u, \text{ or } 2u = t - 1.$$

We shall therefore obtain, by tracing back our substitutions,

$$\begin{aligned} t &= 2u + 1, \\ s &= 3t + u = 7u + 3, \\ r &= s + t = 9u + 4, \\ q &= r + s = 16u + 7, \\ p &= q + r = 25u + 11. \end{aligned}$$

So that the number of women was $25u + 11$, and that of men was $16u + 7$; and in these formulæ we may substitute

* As the numbers 176 and 253 ought, respectively, to be divisible by 39 and 56; and as the former ought, by the question, to leave the remainder 16, and the latter 27, the sum 9883 is formed by multiplying 176 by 56, and adding the remainder 27 to the product: or by multiplying 253 by 39, and adding the remainder 16 to the product. Thus,

$(176 \times 56) + 27 = 9883$; and $(253 \times 39) + 16 = 9883$.

for u any integer numbers whatever. The least results, therefore, will be as follow:

Number of women, 11, 36, 61, 86, 111, &c.
——— of men, 7, 23, 39, 55, 71, &c.

According to the first answer, or that which contains the least numbers, the women expended 176 livres, and the men 175 livres; that is, one livre less than the women.

16. *Question* 10. A person buys some horses and oxen: he pays 31 crowns per horse, and 20 crowns for each ox; and he finds that the oxen cost him 7 crowns more than the horses. How many oxen and horses did he buy?

If we suppose p to be the number of the oxen, and q the number of the horses, we shall have the following equation:

$$p = \frac{31q+7}{20} = q + \frac{11q+7}{20} = q + r, \text{ or } 20r = 11q+7,$$

$$q = \frac{20r-7}{11} = r + \frac{9r-7}{11} = r + s, \text{ or } 11s = 9r-7,$$

$$r = \frac{11s+7}{9} = s + \frac{2s+7}{9} = s + t, \text{ or } 9t = 2s+7,$$

$$s = \frac{9t-7}{2} = 4t + \frac{t-7}{2} = 4t + u, \text{ or } 2u = t-7,$$

whence $t \ldots\ldots = 2u + 7$, and, consequently,

$s = 4t + u = 9u + 28$,
$r = s + t = 11u + 35$,
$q = r + s = 20u + 63$, number of horses,
$p = q + r = 31u + 98$, number of oxen.

Whence, the least positive values of p and q are found by making $u = -3$; those which are greater succeed in the following arithmetical progressions:

Number of oxen, p = 5, 36, 67, 98, 129, 160, 191, 222, 253, &c.

Number of horses, q = 3, 23, 43, 63, 83, 103, 123, 143, 163, &c.

17. If now we consider how the letters p and q, in this example, are determined by the succeeding letters, we shall perceive that this determination depends on the ratio of the numbers 31 and 20, and particularly on the ratio which we discover by seeking the greatest common divisor of these two numbers. In fact, if we perform this operation,

$$\begin{array}{l} 20)\ 31\ (1 \\ \quad\ \ 20 \\ \quad\ \ \overline{11)\ 20\ (1} \\ \qquad\quad\ 11 \\ \qquad\quad\ \overline{9)\ 11\ (1} \\ \qquad\qquad\ \ 9 \\ \qquad\qquad\ \overline{2)\ 9\ (4} \\ \qquad\qquad\quad\ 8 \\ \qquad\qquad\quad\ \overline{1)\ 2\ (2} \\ \qquad\qquad\qquad\ 2 \\ \qquad\qquad\qquad\ \overline{0,} \end{array}$$

it is evident that the quotients are found also in the successive values of the letters p, q, r, s, &c. and that they are connected with the first letter to the right, while the last always remains alone. We see, farther, that the number 7 occurs only in the fifth and last equation, and is affected by the sign $+$, because the number of this equation is odd; for if that number had been even, we should have obtained -7. This will be made more evident by the following Table, in which we may observe the decomposition of the numbers 31 and 20, and then the determination of the values of the letters p, q, r, &c.

$$\begin{array}{r|l} 31 = 1 \times 20 + 11 & p = 1 \times q + r \\ 20 = 1 \times 11 + \ \ 9 & q = 1 \times r + s \\ 11 = 1 \times \ \ 9 + \ \ 2 & r = 1 \times s + t \\ 9 = 4 \times \ \ 2 + \ \ 1 & s = 4 \times t + u \\ 2 = 2 \times \ \ 1 + \ \ 0 & t = 2 \times u + 7. \end{array}$$

18. In the same manner, we may represent the example in Art. 14.

$$\begin{array}{r|l} 56 = 1 \times 39 + 17 & p = 1 \times q + r \\ 39 = 2 \times 17 + \ \ 5 & q = 2 \times r + s \\ 7 = 3 \times \ \ 5 + \ \ 2 & r = 3 \times s + t \\ 5 = 2 \times \ \ 2 + \ \ 1 & s = 2 \times t + u \\ 2 = 2 \times \ \ 1 + \ \ 0 & t = 2 \times u + 11. \end{array}$$

19. And, in the same manner, we may analyse all questions of this kind. For, let there be given the equation $bp = aq + n$, in which a, b, and n, are known numbers; then, we have only to proceed as we should do to find the greatest common divisor of the numbers a and b, and we

may immediately determine p and q by the succeeding letters, as follows:

$$\text{Let} \left\{\begin{array}{l} a = Ab + c \\ b = Bc + d \\ c = Cd + e \\ d = De + f \\ e = Ef + g \\ f = Fg + o \end{array}\right. \text{ and we shall find } \left\{\begin{array}{l} p = Aq + r \\ q = Br + s \\ r = Cs + t \\ s = Dt + u \\ t = Eu + v \\ u = Fv \pm n. \end{array}\right.$$

We have only to observe farther, that in the last equation the sign $+$ must be prefixed to n, when the number of equations is odd; and that, on the contrary, we must take $-n$, when the number is even: by these means, the questions which form the subject of the present chapter may be readily answered, of which we shall give some examples.

20. *Question* 11. Required a number, which, being divided by 11, leaves the remainder 3; but being divided by 19, leaves the remainder 5.

Call this number N; then, in the first place, we have $N = 11p + 3$, and in the second, $N = 19q + 5$; therefore, we have the equation $11p = 19q + 2$, which furnishes the following Table:

$$\begin{array}{r|l} 19 = 1 \times 11 + 8 & p = q + r \\ 11 = 1 \times 8 + 3 & q = r + s \\ 8 = 2 \times 3 + 2 & r = 2s + t \\ 3 = 1 \times 2 + 1 & s = t + u \\ 2 = 2 \times 1 + 0 & t = 2u + 2, \end{array}$$

where we may assign any value to u, and determine by it the preceding letters successively. We find,

$$\begin{array}{l} t \ldots\ldots = 2u + 2 \\ s = t + u = 3u + 2 \\ r = 2s + t = 8u + 6 \\ q = r + s = 11u + 8 \\ p = q + r = 19u + 14; \end{array}$$

whence we obtain the number sought $N = 209u + 157$; therefore 157 is the least number that can express N, or satisfy the terms of the question.

21. *Question* 12. To find a number N such, that if we divide it by 11, there remains 3, and if we divide it by 19, there remains 5; and farther, if we divide it by 29, there remains 10.

The last condition requires that $N = 29p + 10$; and as we have already performed the calculation (in the last question) for the two others, we must, in consequence of that result, have $N = 209u + 157$, instead of which we shall write $N = 209q + 157$; so that

$$29p + 10 = 209q + 157, \text{ or } 29p = 209q + 147;$$

whence we have the following Table;

$$\begin{array}{r} 209 = 7 \times 29 + 6; \\ 29 = 4 \times 6 + 5; \\ 6 = 1 \times 5 + 1; \\ 5 = 5 \times 1 + 0; \end{array} \quad \Bigg| \quad \text{wherefore} \begin{cases} p = 7q + r, \\ q = 4r + s, \\ r = s + t, \\ s = 5t - 147. \end{cases}$$

And, if we now retrace these steps, we have

$$\begin{aligned} s \ldots\ldots\ldots &= 5t - 147, \\ r = s + t &= 6t - 147, \\ q = 4r + s &= 29t - 735, \\ p = 7q + r &= 209t - 5292^*. \end{aligned}$$

So that $N = 6061t - 153458$: and the least number is found by making $t = 26$, which supposition gives $N = 4128$.

22. It is necessary, however, to observe, in order that an equation of the form $bp = aq + n$ may be resolvible, that the two numbers a and b must have no common divisor; for, otherwise, the question would be impossible, unless the number n had the same common divisor.

If it were required, for example, to have $9p = 15q + 2$; since 9 and 15 have a common divisor 3, and which is not a divisor of 2, it is impossible to resolve the question, because $9p - 15q$ being always divisible by 3, can never become $= 2$. But if in this example $n = 3$, or $n = 6$, &c. the question would be posssible: for it would be sufficient first to divide by 3; since we should obtain $3p = 5q + 1$, an equation easily resolvible by the rule already given. It is evident, therefore, that the numbers a, b, ought to have no common divisor, and that our rule cannot apply in any other case.

23. To prove this still more satisfactorily, we shall consider the equation $9p = 15q + 2$ according to the usual method. Here we find

$p = \dfrac{15q+2}{9} = q + \dfrac{6q+2}{9} = q + r$; so that

$9r = 6q + 2$, or $6q = 9r - 2$; or

$q = \dfrac{9r-2}{6} = r + \dfrac{3r-2}{6} = r + s$; so that $3r - 2 = 6s$,

* That is, $-5292 \times 29 = -153468$; to which if the remainder $+10$ required by the question be added, the sum is -153458.

or $3r = 6s + 2$: consequently, $r = \frac{6s+2}{3} = 2s + \frac{2}{3}$.

Now, it is evident, that this can never become an integer number, because s is necessarily an integer; which shews the impossibility of such questions *.

CHAP. II.

Of the Rule *which is called* Regula Cæci, *for determining by means of two* Equations, *three or more* Unknown Quantities.

24. In the preceding chapter, we have seen how, by means of a single equation, two unknown quantities may be determined, so far as to express them in integer and positive numbers. If, therefore, we had two equations, in order that the question may be indeterminate, those equations must contain more than two unknown quantities. Questions of this kind occur in the common books of arithmetic; and are resolved by the rule called *Regula Cæci*, *Position*, or *The Rule of False*; the foundation of which we shall now explain, beginning with the following example:

25. *Question* 1. Thirty persons, men, women, and children, spend 50 crowns in a tavern; the share of a man is 3 crowns, that of a woman 2 crowns, and that of a child is 1 crown; how many persons were there of each class?

If the number of men be p, of women q, and of children r, we shall have the two following equations;

$$1.\ p + q + r = 30, \text{ and}$$
$$2.\ 3p + 2q + r = 50,$$

from which it is required to find the value of the three letters p, q, and r, in integer and positive numbers. The first equation gives $r = 30 - p - q$; whence we immediately conclude that $p + q$ must be less than 30; and, substituting this value of r in the second equation, we have $2p + q + 30 = 50$; so that $q = 20 - 2p$, and $p + q =$

* See the Appendix to this chapter, at Art. 3. of the Additions by De la Grange. p. 530.