This formula contains a general rule for finding all the possible polygonal roots of given numbers.

For example, let there be given the XXIV-gonal number, 3009: since a is here $= 3009$ and $n = 24$, we have $n - 2 = 22$ and $n - 4 = 20$; wherefore the root, or

$$x, = \frac{20+\sqrt{(529584+400)}}{44} = \frac{20+728}{44} = 17.$$

CHAP. VIII.

Of the Extraction *of the* Square Roots *of* Binomials.

669. By a *binomial** we mean a quantity composed of two parts, which are either both affected by the sign of the square root, or of which one, at least, contains that sign.

For this reason $3 + \sqrt{5}$ is a binomial, and likewise $\sqrt{8} + \sqrt{3}$; and it is indifferent whether the two terms be joined by the sign + or by the sign −. So that $3 - \sqrt{5}$ and $3 + \sqrt{5}$ are both binomials.

670. The reason that these binomials deserve particular attention is, that in the resolution of quadratic equations we are always brought to quantities of this form, when the resolution cannot be performed. For example, the equation $x^2 = 6x - 4$ gives $x = 3 + \sqrt{5}$.

It is evident, therefore, that such quantities must often occur in algebraic calculations; for which reason, we have already carefully shewn how they are to be treated in the ordinary operations of addition, subtraction, multiplication, and division. but we have not been able till now to shew how their square roots are to be extracted; that is, so far as that extraction is possible; for when it is not, we must be satisfied with affixing to the quantity another radical sign. Thus, the square root of $3 + \sqrt{2}$ is written $\sqrt{\overline{3 + \sqrt{2}}}$; or $\sqrt{(3 + \sqrt{2})}$.

671. It must here be observed, in the first place, that the

* In algebra we generally give the name *binomial* to any quantity composed of two terms; but Euler has thought proper to confine this appellation to those expressions, which the French analysts call *quantities partly commensurable, and partly incommensurable.* F. T.

squares of such binomials are also binomials of the same kind; in which also one of the terms is always rational.

For, if we take the square of $a + \surd b$, we shall obtain $(a^2 + b) + 2a \surd b$. If therefore it were required reciprocally to take the root of the quantity $(a^2 + b) + 2a \surd b$, we should find it to be $a + \surd b$, and it is undoubtedly much easier to form an idea of it in this manner, than if we had only put the sign $\surd$ before that quantity. In the same manner, if we take the square of $\surd a + \surd b$, we find it $(a + b) + 2 \surd ab$; therefore, reciprocally, the square root of $(a + b) + 2 \surd ab$ will be $\surd a + \surd b$, which is likewise more easily understood, than if we had been satisfied with putting the sign $\surd$ before the quantity.

672. It is chiefly required, therefore, to assign a character, which may, in all cases, point out whether such a square root exists or not; for which purpose we shall begin with an easy quantity, requiring whether we can assign, in the sense that we have explained, the square root of the binomial $5 + 2 \surd 6$.

Suppose, therefore, that this root is $\surd x + \surd y$; the square of it is $(x + y) + 2 \surd xy$, which must be equal to the quantity $5 + 2 \surd 6$. Consequently, the rational part $x + y$ must be equal to 5, and the irrational part $2 \surd xy$ must be equal to $2 \surd 6$; which last equality gives $\surd xy = \surd 6$. Now, since $x + y = 5$, we have $y = 5 - x$, and this value substituted in the equation $xy = 6$, produces $5x - x^2 = 6$, or $x^2 = 5x - 6$; therefore, $x = \frac{5}{2} + \surd(\frac{25}{4} - \frac{24}{4}) = \frac{5}{2} + \frac{1}{2} = 3$. So that $x = 3$ and $y = 2$; whence we conclude, that the square root of $5 + 2 \surd 6$ is $\surd 3 + \surd 2$.

673. As we have here found the two equations, $x + y = 5$, and $xy = 6$, we shall give a particular method for obtaining the values of x and y.

Since $x + y = 5$, by squaring, $x^2 + 2xy + y^2 = 25$; and as we know that $x^2 - 2xy + y^2$ is the square of $x - y$, let us subtract from $x^2 + 2xy + y^2 = 25$, the equation $xy = 6$, taken four times, or $4xy = 24$, in order to have $x^2 - 2xy + y^2 = 1$; whence by extraction we have $x - y = 1$; and as $x + y = 5$, we shall easily find $x = 3$, and $y = 2$: wherefore, the square root of $5 + 2 \surd 6$ is $\surd 3 + \surd 2$.

674. Let us now consider the general binomial $a + \surd b$, and supposing its square root to be $\surd x + \surd y$, we shall have the equation $(x + y) + 2 \surd xy = a + \surd b$; so that $x + y = a$, and $2 \surd xy = \surd b$, or $4xy = b$; subtracting this square from the square of the equation $x + y = a$, that is, from $x^2 + 2xy + y^2 = a^2$, there remains $x^2 - 2xy + y^2 = a^2 - b$, the square root of which is $x - y = \surd(a^2 - b)$.

Now, $x + y = a$; we have therefore $x = \frac{a + \surd(a^2 - b)}{2}$,

and $y = \frac{a - \surd(a^2 - b)}{2}$; consequently, the square root required of $a + \surd b$ is $\surd\frac{(a + \surd(a^2 - b))}{2} + \surd\frac{(a - \surd(a^2 - b))}{2}$.

675. We admit that this expression is more complicated than if we had simply put the radical sign $\surd$ before the given binomial $a + \surd b$, and written it $\surd(a + \surd b)$: but the above expression may be greatly simplified when the numbers a and b are such, that $a^2 - b$ is a square; since then the sign $\surd$, which is under the radical, disappears. We see also, at the same time, that the square root of the binomial $a + \surd b$ cannot be conveniently extracted, except when $a^2 - b = c^2$; for in this case the square root required is $\surd(\frac{a + c}{2}) + \surd(\frac{a - c}{2})$: but if $a^2 - b$ is not a perfect square, we cannot express the square root of $a + \surd b$ more simply, than by putting the radical sign $\surd$ before it.

676. The condition, therefore, which is requisite, in order that we may express the square root of a binomial $a + \surd b$ in a more convenient form, is, that $a^2 - b$ be a square; and if we represent that square by c^2, we shall have for the square root in question $\surd(\frac{a + c}{2}) + \surd(\frac{a - c}{2})$. We must farther remark, that the square root of $a - \surd b$ will be $\surd(\frac{a + c}{2}) - \surd(\frac{a - c}{2})$; for, by squaring this quantity, we get $a - 2\surd(\frac{a^2 - c^2}{4})$; now, since $c^2 = a^2 - b$, and consequently $a^2 - c^2 = b$, the same square is found

$$= a - 2\surd\frac{b}{4} = a - \frac{2\surd b}{2} = a - \surd b.$$

677. When it is required, therefore, to extract the square root of a binomial, as $a \pm \surd b$, the rule is, to subtract from the square a^2 of the rational part the square b of the irrational part, to take the square root of the remainder, and calling that root c, to write for the root required

$$\surd(\frac{a + c}{2}) \pm \surd(\frac{a - c}{2}).$$

678. If the square root of $2 + \surd 3$ were required, we

should have $a = 2$ and $\sqrt{b} = \sqrt{3}$; wherefore $a^2 - b = c^2 = 1$; so that, by the formula just given, the root sought $= \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}$.

Let it be required to find the square root of the binomial $11 + 6\sqrt{2}$. Here we shall have $a = 11$, and $\sqrt{b} = 6\sqrt{2}$; consequently, $b = 36 \times 2 = 72$, and $a^2 - b = 49$, which gives $c = 7$; and hence we conclude, that the square root of $11 + 6\sqrt{2}$ is $\sqrt{9} + \sqrt{2}$, or $3 + \sqrt{2}$.

Required the square root of $11 + 2\sqrt{30}$. Here $a = 11$, and $\sqrt{b} = 2\sqrt{30}$; consequently, $b = 4 \times 30 = 120$, $a^2 - b = 1$, and $c = 1$; therefore the root required is $\sqrt{6} + \sqrt{5}$.

679. This rule also applies, even when the binomial contains imaginary, or impossible quantities.

Let there be proposed, for example, the binomial $1 + 4\sqrt{-3}$. First, we shall have $a = 1$ and $\sqrt{b} = 4\sqrt{-3}$, that is to say, $b = -48$, and $a^2 - b = 49$; therefore $c = 7$, and consequently the square root required is $\sqrt{4} + \sqrt{-3} = 2 + \sqrt{-3}$.

Again, let there be given $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$. First, we have $a = -\frac{1}{2}$; $\sqrt{b} = \frac{1}{2}\sqrt{-3}$, and $b = \frac{1}{4} \times -3 = -\frac{3}{4}$; whence $a^2 - b = \frac{1}{4} + \frac{3}{4} = 1$, and $c = 1$; and the result required is $\sqrt{\frac{1}{4}} + \sqrt{-\frac{3}{4}} = \frac{1}{2} + \frac{\sqrt{-3}}{2}$, or $\frac{1}{2} + \frac{1}{2}\sqrt{-3}$.

Another remarkable example is that in which it is required to find the square root of $2\sqrt{-1}$. As there is here no rational part, we shall have $a = 0$. Now, $\sqrt{b} = 2\sqrt{-1}$, and $b = -4$; wherefore $a^2 - b = 4$ and $c = 2$; consequently, the square root required is $\sqrt{1} + \sqrt{-1} = 1 + \sqrt{-1}$, and the square of this quantity is found to be $1 + 2\sqrt{-1} - 1 = 2\sqrt{-1}$.

680. Suppose now we have such an equation as $x^2 = a \pm \sqrt{b}$, and that $a^2 - b = c^2$; we conclude from this, that the value of $x = \sqrt{(\frac{a+c}{2})} \pm \sqrt{(\frac{a-c}{2})}$, which may be useful in many cases.

For example, if $x^2 = 17 + 12\sqrt{2}$, we shall have $x = 3 + \sqrt{8} = 3 + 2\sqrt{2}$.

681. This case occurs most frequently in the resolution of equations of the fourth degree, such as $x^4 = 2ax^2 + d$. For, if we suppose $x^2 = y$, we have $x^4 = y^2$, which reduces the given equation to $y^2 = 2ay + d$, and from this we find $y = a \pm \sqrt{(a^2 + d)}$, therefore, $x^2 = a \pm \sqrt{(a^2 + d)}$, and consequently we have another evolution to perform. Now,

since $\sqrt{b} = \sqrt{(a^2 + d)}$, we have $b = a^2 + d$, and $a^2 - b = -d$; if, therefore, $-d$ is a square, as c^2, that is to say, $d = -c^2$, we may assign the root required.

Suppose, in reality, that $d = -c^2$; or that the proposed equation of the fourth degree is $x^4 = 2ax^2 - c^2$, we shall then find that $x = \sqrt{(\frac{a+c}{2})} \pm \sqrt{(\frac{a-c}{2})}$.

682. We shall illustrate what we have just said by some examples.

1. Required two numbers, whose product may be 105, and whose squares may together make 274.

Let us represent those two numbers by x and y; we shall then have the two equations,

$$xy = 105$$
$$x^2 + y^2 = 274.$$

The first gives $y = \frac{105}{x}$, and this value of y being substituted in the second equation, we have

$$x^2 + \frac{105^2}{x^2} = 274.$$

Wherefore $x^4 + 105^2 = 274x^2$, or $x^4 = 274x^2 - 105^2$.

If we now compare this equation with that in the preceding article, we have $2a = 274$, and $-c^2 = -105^2$; consequently, $c = 105$, and $a = 137$. We therefore find

$$x = \sqrt{(\frac{137+105}{2})} \pm \sqrt{(\frac{137-105}{2})} = 11 \pm 4.$$

Whence $x = 15$, or $x = 7$. In the first case, $y = 7$, and in the second case, $y = 15$; whence the two numbers sought are 15 and 7.

683. It is proper, however, to observe, that this calculation may be performed much more easily in another way. For, since $x^2 + 2xy + y^2$ and $x^2 - 2xy + y^2$ are squares, and since the values of $x^2 + y^2$ and of xy are given, we have only to take the double of this last quantity, and then to add and subtract it from the first, as follows: $x^2 + y^2 = 274$; to which if we add $2xy = 210$, we have
$x^2 + 2xy + y^2 = 484$, which gives $x + y = 22$.

But subtracting $2xy$, there remains $x^2 - 2xy + y^2 = 64$, whence we find $x - y = 8$.

So that $2x = 30$, and $2y = 14$; consequently, $x = 15$ and $y = 7$.

The following general question is resolved by the same method.

2. Required two numbers, whose product may be m, and the sum of the squares n.

If those numbers are represented by x and y, we have the two following equations:

$$xy = m$$
$$x^2 + y^2 = n.$$

Now, $2xy = 2m$ being added to $x^2 + y^2 - n$, we have $x^2 + 2xy + y^2 = n + 2m$, and consequently,

$$x + y = \surd(n + 2m).$$

But subtracting $2xy$, there remains $x^2 - 2xy + y^2 = n - 2m$, whence we get $x - y = \surd(n - 2m)$; we have, therefore, $x = \frac{1}{2}\surd(n + 2m) + \frac{1}{2}\surd(n - 2m)$; and

$$y = \tfrac{1}{2}\surd(n + 2m) - \tfrac{1}{2}\surd(n - 2m).$$

684. 3. Required two numbers, such, that their product may be 35, and the difference of their squares 24.

Let the greater of the two numbers be x, and the less y: then we shall have the two equations

$$xy = 35,$$
$$x^2 - y^2 = 24;$$

and as we have not the same advantages here, we shall proceed in the usual manner. Here, the first equation gives $y = \frac{35}{x}$, and, substituting this value of y in the second, we have $x^2 - \frac{1225}{x^2} = 24$. Multiplying by x^2, we have $x^4 - 1225 = 24x^2$; or $x^4 = 24x^2 + 1225$. Now, the second member of this equation being affected by the sign $+$, we cannot make use of the formula already given, because having $c^2 = -1225$, c would become imaginary.

Let us therefore make $x^2 = z$; we shall then have $z^2 = 24z + 1225$, whence we obtain

$$z = 12 \pm \surd(144 + 1225) \text{ or } z = 12 \pm 37;$$

consequently, $x^2 = 12 \pm 37$; that is to say, either $= 49$, or $= -25$.

If we adopt the first value, we have $x = 7$ and $y = 5$.

The second value gives $x = \surd -25$; and, since $xy = 35$, we have $y = \frac{35}{\surd -25} = \surd\frac{1225}{-25} = \surd -49$.

685. We shall conclude this chapter with the following question.

4. Required two numbers, such, that their sum, their product, and the difference of their squares, may be all equal.

Let x be the greater of the two numbers, and y the less; then the three following expressions must be equal to one another: namely, the sum, $x + y$; the product, xy; and the difference of the squares, $x^2 - y^2$. If we compare the first with the second, we have $x + y = xy$; which will give a value of x: for $y = xy - x = (y - 1)$, and $x = \frac{y}{y-1}$; Consequently, $x + y = \frac{y}{y-1} + y = \frac{y^2}{y-1}$, and $xy = \frac{y^2}{y-1}$, that is to say, the sum is equal to the product; and to this also the difference of the squares ought to be equal. Now, we have $x^2 - y^2 = \frac{y^2}{y^2 - 2y + 1} - y^2 = \frac{-y^4 + 2y^3}{y^2 - 2y + 1}$; so that making this equal to the quantity found $\frac{y^2}{y-1}$, we have $\frac{y^2}{y-1} = \frac{-y^4 + 2y^3}{y^2 - 2y + 1}$; dividing by y^2, we have $\frac{1}{y-1} = \ldots\ldots \frac{-y^2 + 2y}{y^2 - 2y + 1}$; and multiplying by $y^2 - 2y + 1$, or $(y - 1)^2$, we have $y - 1 = -y^2 + 2y$; consequently, $y^2 = y + 1$; which gives $y = \frac{1}{2} \pm \sqrt{(\frac{1}{4} + 1)} = \frac{1}{2} \pm \sqrt{\frac{5}{4}}$; or $y = \frac{1 \pm \sqrt{5}}{2}$, and since $x = \frac{y}{y-1}$, we shall have, by substitution, and using the sign $+$, $x = \frac{\sqrt{5}+1}{\sqrt{5}-1}$.

In order to remove the surd quantity from the denominator, multiply both terms by $\sqrt{5} + 1$, and we obtain $x = \frac{6 + 2\sqrt{5}}{4} = \frac{3 + \sqrt{5}}{2}$.

Therefore the greater of the numbers sought, or x, $= \frac{3 + \sqrt{5}}{2}$; and the less, y, $= \frac{1 + \sqrt{5}}{2}$.

Hence their sum $x + y = 2 + \sqrt{5}$; their product $xy = 2 + \sqrt{5}$; and since $x^2 = \frac{7 + 3\sqrt{5}}{2}$, and $y^2 = \frac{3 + \sqrt{5}}{2}$, we have also the difference of the squares $x^2 - y^2 = 2 + \sqrt{5}$, being all the same quantity.

686. As this solution is very long, it is proper to remark

that it may be abridged. In order to which, let us begin with making the sum $x+y$ equal to the difference of the squares x^2-y^2; we shall then have $x+y=x^2-y^2$; and dividing by $x+y$, because $x^2-y^2=(x+y)\times(x-y)$, we find $1=x-y$ and $x=y+1$. Consequently, $x+y=2y+1$, and $x^2-y^2=2y+1$; farther, as the product xy, or y^2+y, must be equal to the same quantity, we have $y^2+y=2y+1$, or $y^2=y+1$, which gives, as before, $y=\frac{1+\sqrt{5}}{2}$.

687. The preceding question leads also to the solution of the following.

5. To find two numbers, such, that their sum, their product, and the sum of their squares, may be all equal.

Let the numbers sought be represented by x and y; then there must be an equality between $x+y$, xy, and x^2+y^2.

Comparing the first and second quantities, we have $x+y=xy$, whence $x=\frac{y}{y-1}$; consequently, xy, and $x+y=\frac{y^2}{y-1}$. Now, the same quantity is equal to x^2+y^2; so that we have

$$\frac{y^2}{y^2-2y+1}+y^2=\frac{y^2}{y-1}.$$

Multiplying by y^2-2y+1, the product is

$$y^4-2y^3+2y^2=y^3-y^2, \text{ or } y^4=3y^3-3y^2;$$

and dividing by y^2, we have $y^2=3y-3$; which gives $y=\frac{3}{2}\pm\sqrt{(\frac{9}{4}-3)}=\frac{3+\sqrt{-3}}{2}$; consequently, $y-1=\frac{1+\sqrt{-3}}{2}$, whence results $x=\frac{3+\sqrt{-3}}{1+\sqrt{-3}}$; and multiplying both terms by $1-\sqrt{-3}$, the result is

$$x=\frac{6-2\sqrt{-3}}{4}, \text{ or } \frac{3-\sqrt{-3}}{2}.$$

Therefore the numbers sought are $x=\frac{3-\sqrt{-3}}{2}$, and $y=\frac{3+\sqrt{-3}}{2}$, the sum of which is $x+y=3$, their product $xy=3$; and lastly, since $x^2=\frac{3-3\sqrt{-3}}{2}$, and

$y^2 = \frac{3+3\sqrt{-3}}{2}$, the sum of the squares $x^2 + y^2 = 3$, all the same quantity as required.

688. We may greatly abridge this calculation by a particular artifice, which is applicable likewise to other cases; and which consists in expressing the numbers sought by the sum and the difference of two letters, instead of representing them by distinct letters.

In our last question, let us suppose one of the numbers sought to be $p+q$, and the other $p-q$, then their sum will be $2p$, their product will be p^2-q^2, and the sum of their squares will be $2p^2+2q^2$, which three quantities must be equal to each other; therefore making the first equal to the second, we have $2p = p^2 - q^2$, which gives $q^2 = p^2 - 2p$.

Substituting this value of q^2 in the third quantity $(2p^2+2q^2)$, and comparing the result $4p^2-4p$ with the first, we have $2p = 4p^2 - 4p$, whence $p = \frac{3}{2}$.

Consequently, $q^2 = p^2 - 2p = -\frac{3}{4}$, and $q = \frac{\sqrt{-3}}{2}$;

so that the numbers sought are $p+q = \frac{3+\sqrt{-3}}{2}$, and

$p-q = \frac{3-\sqrt{-3}}{2}$, as before.

QUESTIONS FOR PRACTICE.

1. What two numbers are those, whose difference is 15, and half of their product equal to the cube of the less?

Ans. 3 and 18.

2. To find two numbers whose sum is 100, and product 2059. *Ans.* 71 and 29.

3. There are three numbers in geometrical progression: the sum of the first and second is 10, and the difference of the second and third is 24. What are they?

Ans. 2, 8 and 32.

4. A merchant having laid out a certain sum of money in goods, sells them again for 24*l.* gaining as much per cent as the goods cost him: required, what they cost him. *Ans.* 20*l.*

5. The sum of two numbers is a, their product b. Required the numbers.

Ans. $\frac{a}{2} \pm \sqrt{(-b + \frac{a^2}{4})}$, and

$\frac{a}{2} \mp \sqrt{(-b + \frac{a^2}{4})}$.

6. The sum of two numbers is a, and the sum of their squares b. Required the numbers.

$$Ans.\quad \frac{a}{2} \pm \sqrt{(\frac{2b-a^2}{4})}, \text{ and}$$

$$\frac{a}{2} \mp \sqrt{(\frac{2b-a^2}{4})}.$$

7. To divide 36 into three such parts, that the second may exceed the first by 4, and that the sum of all their squares may be 464. *Ans.* 8, 12, 16.

8. A person buying 120 pounds of pepper, and as many of ginger, finds that for a crown he has one pound more of ginger than of pepper. Now, the whole price of the pepper exceeded that of the ginger by six crowns: how many pounds of each had he for a crown?

Ans. 4 of pepper, and 5 of ginger.

9. Required three numbers in continual proportion, 60 being the middle term, and the sum of the extremes being equal to 125. *Ans.* 45, 60, 80.

10. A person bought a certain number of oxen for 80 guineas: if he had received 4 more for the same money, he would have paid one guinea less for each head. What was the number of oxen? *Ans.* 16.

11. To divide the number 10 into two such parts, that their product being added to the sum of their squares, may make 76. *Ans.* 4 and 6.

12. Two travellers A and B set out from two places, Γ and Δ, at the same time; A from Γ with a design to pass through Δ, and B from Δ to travel the same way: after A had overtaken B, they found on computing their travels, that they had both together travelled 30 miles; that A had passed through Δ four days before, and that B, at his rate of travelling, was a journey of nine days distant from Γ. Required the distance between the places Γ and Δ. *Ans.* 6 miles.