$$x^4 - 6x^3 + 12x^2 - 12x + 4 = 0,$$

which must be contained in the formula

$$(x^2 - 3x + p)^2 - (qx + r)^2 = 0,$$

in the former part of which we have put $-3x$, because -3 is half the coefficient -6, of the given equation. This formula being expanded, gives

$$x^4 - 6x^3 + (2p + 9 - q^2)x^2 - (6p + 2qr)x + p^2 - r^2 = 0;$$

which, compared with our equation, there will result from that comparison the following equations:

$$1.\ 2p + 9 - q^2 = 12,$$
$$2.\ 6p + 2qr = 12,$$
$$3.\ p^2 - r^2 = 4.$$

The first gives $q^2 = 2p - 3$;
the second, $2qr = 12 - 6p$, or $qr = 6 - 3p$;
the third, $r^2 = p^2 - 4$.
Multiplying r^2 by q^2, and $p^2 - 4$ by $2p - 3$, we have

$$q^2r^2 = 2p^3 - 3p^2 - 8p + 12;$$

and if we square the value of qr, we have

$$q^2r^2 = 36 - 36p + 9p^2;$$

so that we have the equation

$$2p^3 - 3p^2 - 8p + 12 = 9p^2 - 36p + 36, \text{ or}$$
$$2p^3 - 12p^2 + 28p - 24 = 0, \text{ or}$$
$$p^3 - 6p^2 + 14p - 12 = 0,$$

one of the roots of which is $p = 2$; and it follows that $q^2 = 1$, $q = 1$, and $qr - r = 0$. Therefore our equation will be $(x^2 - 3x + 2)^2 = x^2$, and its square root will be $x^2 - 3x + 2 = \pm x$. If we take the upper sign, we have $x^2 = 4x - 2$; and taking the lower sign, we obtain $x^2 = 2x - 2$, whence we derive the four roots $x = 2 \pm \sqrt{2}$, and $x = 1 \pm \sqrt{-1}$.

CHAP. XV.

Of a new Method *of resolving* Equations *of the* Fourth Degree.

773. The rule of Bombelli, as we have seen, resolves equations of the fourth degree by means of an equation of the third degree; but since the invention of that Rule,

another method has been discovered of performing the same resolution: and, as it is altogether different from the first, it deserves to be separately explained *.

774. We suppose that the root of an equation of the fourth degree has the form, $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$, in which the letters p, q, r, express the roots of an equation of the third degree, $z^3 - fz^2 + gz - h = 0$; so that $p + q + r = f$; $pq + pr + qr = g$; and $pqr = h$. This being laid down, we square the assumed formula, $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$, and we obtain

$$x^2 = p + q + r + 2\sqrt{pq} + 2\sqrt{pr} + 2\sqrt{qr};$$

and, since $p + q + r = f$, we have

$$x^2 - f = 2\sqrt{pq} + 2\sqrt{pr} + 2\sqrt{qr};$$

we again take the squares, and find

$$x^4 - 2fx^2 + f^2 = 4pq + 4pr + 4qr + 8\sqrt{p^2qr} + 8\sqrt{pq^2r} + 8\sqrt{pqr^2}.$$

Now, $4pq + 4pr + 4qr = 4g$; so that the equation becomes $x^4 - 2fx^2 + f^2 - 4g = 8\sqrt{pqr} \times (\sqrt{p} + \sqrt{q} + \sqrt{r})$; but $\sqrt{p} + \sqrt{q} + \sqrt{r} = x$, and $pqr = h$, or $\sqrt{pqr} = \sqrt{h}$; wherefore we arrive at this equation of the fourth degree, $x^4 - 2fx^2 - 8x\sqrt{h} + f^2 - 4g = 0$, one of the roots of which is $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$; and in which p, q, and r, are the roots of an equation of the third degree,

$$z^3 - fz^2 + gz - h = 0.$$

775. The equation of the fourth degree, at which we have arrived, may be considered as general, although the second term x^3y is wanting; for we shall afterwards shew, that every complete equation may be transformed into another, from which the second term has been taken away.

Let there be proposed the equation $x^4 - ax^2 - bx - c = 0$, in order to determine its root. This we must first compare with the formula, in order to obtain the values of f, g, and h; and we shall have,

1. $2f = a$, and, consequently, $f = \frac{a}{2}$;

2. $8\sqrt{h} = b$, so that $h = \frac{b^2}{64}$;

3. $f^2 - 4g = -c$, or $\frac{a^2}{4} - 4g + c = 0$,

or $\frac{1}{4}a^2 + c = 4g$; consequently, $g = \frac{1}{16}a^2 + \frac{1}{4}c$.

* This method was the invention of Euler himself. He has explained it in the sixteenth volume of the Ancient Commentaries of Petersburg. F. T.

776. Since, therefore, the equation

$$x^4 - ax^2 - bx - c = 0,$$

gives the values of the letters f, g, and h, so that

$$f = \tfrac{1}{2}a,\ g = \tfrac{1}{16}a^2 + \tfrac{1}{4}c, \text{ and } h = \tfrac{1}{64}b^2, \text{ or } \sqrt{h} = \tfrac{1}{8}b,$$

we form from these values the equation of the third degree $z^3 - fz^2 + gz - h = 0$, in order to obtain its roots by the known rule. And if we suppose those roots, 1. $z = p$, 2. $z = q$, 3. $z = r$, one of the roots of our equation of the fourth degree must be, by the supposition, Art. 774,

$$x = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

777. This method appears at first to furnish only one root of the given equation; but if we consider that every sign $\sqrt{\ }$ may be taken negatively, as well as positively, we shall immediately perceive that this formula contains all the four roots.

Farther, if we chose to admit all the possible changes of the signs, we should have eight different values of x, and yet four only can exist. But it is to be observed, that the product of those three terms, or $\sqrt{pqr}$, must be equal to $\sqrt{h} = \frac{1}{8}b$, and that if $\frac{1}{8}b$ be positive, the product of the terms $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, must likewise be positive; so that all the variations that can be admitted are reduced to the four following:

$$\begin{aligned} 1.\ x &= \sqrt{p} + \sqrt{q} + \sqrt{r}, \\ 2.\ x &= \sqrt{p} - \sqrt{q} - \sqrt{r}, \\ 3.\ x &= -\sqrt{p} + \sqrt{q} - \sqrt{r}. \\ 4.\ x &= -\sqrt{p} - \sqrt{q} + \sqrt{r}. \end{aligned}$$

In the same manner, when $\frac{1}{8}b$ is negative, we have only the four following values of x:

$$\begin{aligned} 1.\ x &= \sqrt{p} + \sqrt{q} - \sqrt{r}, \\ 2.\ x &= \sqrt{p} - \sqrt{q} + \sqrt{r}, \\ 3.\ x &= -\sqrt{p} + \sqrt{q} + \sqrt{r}, \\ 4.\ x &= -\sqrt{p} - \sqrt{q} - \sqrt{r}. \end{aligned}$$

This circumstance enables us to determine the four roots in all cases; as may be seen in the following example.

778. Let there be proposed the equation of the fourth degree, $x^4 - 25x^2 + 60x - 36 = 0$, in which the second term is wanting. Now, if we compare this with the general formula, we have $a = 25$, $b = -60$, and $c = 36$; and after that,

$$f = \tfrac{25}{2},\ g = \tfrac{625}{16} + 9 = \tfrac{769}{16}, \text{ and } h = \tfrac{225}{4};$$

by which means our equation of the third degree becomes,

$$z^3 - \tfrac{25}{2}z^2 + \tfrac{769}{16}z - \tfrac{225}{4} = 0.$$

First, to remove the fractions, let us make $z = \frac{u}{4}$; and we shall have $\frac{u^3}{64} - \frac{25u^2}{32} + \frac{769u}{64} - \frac{225}{4} = 0$, and multiplying by the greatest denominator, we obtain

$$u^3 - 50u^2 + 769u - 3600 = 0.$$

We have now to determine the three roots of this equation; which are all three found to be positive; one of them being $u = 9$: then dividing the equation by $u - 9$, we find the new equation $u^2 - 41u + 400 = 0$, or $u^2 = 41u - 400$, which gives

$$u = \tfrac{41}{2} \pm \sqrt{(\tfrac{1681}{4} - \tfrac{1600}{4}} = \frac{41 \pm 9}{2};$$

so that the three roots are $u = 9$, $u = 16$, and $u = 25$.

Consequently, as $z = \frac{u}{4}$ the roots are

$$1.\ z = \tfrac{9}{4},\ 2.\ z = 4,\ 3.\ z = \tfrac{25}{4}.$$

These, therefore, are the values of the letters p, q, and r; that is to say, $p = \frac{9}{4}$, $q = 4$, and $r = \frac{25}{4}$. Now, if we consider that $\sqrt{pqr} = \sqrt{h} = -\frac{15}{2}$, and that therefore this value $= \frac{1}{8}b$ is negative, we must, agreeably to what has been said with regard to the signs of the roots $\sqrt{p}$, $\sqrt{q}$, and $\sqrt{r}$, take all those three roots negatively, or take only one of them negatively; and consequently, as $\sqrt{p} = \frac{3}{2}$, $\sqrt{q} = 2$, and $\sqrt{r} = \frac{5}{2}$, the four roots of the given equation are found to be:

$$\begin{aligned}
1.\ x &= \tfrac{3}{2} + 2 - \tfrac{5}{2} = 1,\\
2.\ x &= \tfrac{3}{2} - 2 + \tfrac{5}{2} = 2,\\
3.\ x &= -\tfrac{3}{2} + 2 + \tfrac{5}{2} = 3,\\
4.\ x &= -\tfrac{3}{2} - 2 - \tfrac{5}{2} = -6.
\end{aligned}$$

From these roots are formed the four factors,

$$(x - 1) \times (x - 2) \times (x - 3) \times (x + 6) = 0.$$

The first two, multiplied together, give $x^2 - 3x + 2$; the product of the last two is $x^2 + 3x - 18$; again multiplying these two products together, we obtain exactly the equation proposed.

779. It remains now to shew how an equation of the fourth degree, in which the second term is found, may be transformed into another, in which that term is wanting: for which we shall give the following rule *.

* An investigation of this rule may be seen in Maclaurin's Algebra, Part II. chap. 3.

Let there be proposed the general equation $y^4 + ay^3 + by^2 + cy + d = 0$. If we add to y the fourth part of the coefficient of the second term, or $\frac{1}{4}a$, and write, instead of the sum, a new letter x, so that $y + \frac{1}{4}a = x$, and consequently $y = x - \frac{1}{4}a$: we shall have

$y^2 = x^2 - \frac{1}{2}ax + \frac{1}{16}a^2$, $y^3 = x^3 - \frac{3}{4}ax^2 + \frac{3}{16}a^2x - \frac{1}{64}a^3$,
and, lastly, as follows:

$$\begin{array}{rlllll}
y^4 = & x^4 - ax^3 & + \frac{3}{8}a^2x^2 & - \frac{1}{16}a^3x & + \frac{1}{256}a^4 \\
ay^3 = & ax^3 & - \frac{3}{4}a^2x^2 & + \frac{3}{16}a^3x & - \frac{1}{64}a^4 \\
by^2 = & & bx^2 & - \frac{1}{2}abx & + \frac{1}{16}a^2b \\
cy = & & & cx & - \frac{1}{4}ac \\
d = & & & & d
\end{array}$$

And hence, by addition,

$$\left.\begin{array}{rlll}
x^4 + 0 & - \frac{3}{8}a^2x^2 & + \frac{1}{8}a^3x & - \frac{3}{256}a^4 \\
 & bx^2 & - \frac{1}{2}abx & + \frac{1}{16}a^2b \\
 & & cx & - \frac{1}{4}ac \\
 & & & d
\end{array}\right\} = 0.$$

We have now an equation from which the second term is taken away, and to which nothing prevents us from applying the rule before given for determining its four roots. After the values of x are found, those of y will easily be determined, since $y = x - \frac{1}{4}a$.

780. This is the greatest length to which we have yet arrived in the resolution of algebraic equations. All the pains that have been taken in order to resolve equations of the fifth degree, and those of higher dimensions, in the same manner, or, at least, to reduce them to inferior degrees, have been unsuccessful: so that we cannot give any general rules for finding the roots of equations, which exceed the fourth degree.

The only success that has attended these attempts has been the resolution of some particular cases; the chief of which is that, in which a rational root takes place; for this is easily found by the method of divisors, because we know that such a root must be always a factor of the last term. The operation, in other respects, is the same as that we have explained for equations of the third and fourth degree.

781. It will be necessary, however, to apply the rule of Bombelli to an equation which has no rational roots.

Let there be given the equation $y^4 - 8y^3 + 14y^2 + 4y - 8 = 0$. Here we must begin with destroying the second term, by adding the fourth of its coefficient to y, supposing $y - 2 = x$, and substituting in the equation, instead of y, its new value $x + 2$, instead of y^2, its value $x^2 + 4x + 4$; and doing the same with regard to y^3 and y^4, we shall have,

$$
\begin{array}{rl}
y^4 = & x^4 + 8x^3 + 24x^2 + 32x + 16 \\
-8y^3 = & \quad -8x^3 - 48x^2 - 96x - 64 \\
14y^2 = & \qquad\qquad 14x^2 + 56x + 56 \\
4y = & \qquad\qquad\qquad\quad 4x + 8 \\
-8 = & \qquad\qquad\qquad\qquad\quad -8 \\
\hline
 & x^4 + 0 - 10x^2 - 4x + 8 = 0.
\end{array}
$$

This equation being compared with our general formula, gives $a = 10$, $b = 4$, $c = -8$; whence we conclude, that $f = 5$, $g = \frac{17}{4}$, $h = \frac{1}{4}$, and $\sqrt{h} = \frac{1}{2}$; that the product $\sqrt{pqr}$ will be positive; and that it is from the equation of the third degree, $z^3 - 5z^2 + \frac{17}{4}z - \frac{1}{4} = 0$, that we are to seek for the three roots p, q, r. (Art. 774.)

782. Let us first remove the fractions from this equation, by making $z = \frac{u}{2}$, and we shall thus have, after multiplying by 8, the equation $u^3 - 10u^2 + 17u - 2 = 0$, in which all the roots are positive. Now, the divisors of the last term are 1 and 2; if we try $u = 1$, we find $1 - 10 + 17 - 2 = 6$; so that the equation is not reduced to nothing: but trying $u = 2$, we find $8 - 40 + 34 - 2 = 0$, which answers to the equation, and shews that $u = 2$ is one of the roots. The two others will be found by dividing by $u - 2$, as usual; then the quotient $u^2 - 8u + 1 = 0$ will give $u^2 = 8u - 1$, and $u = 4 \pm \sqrt{15}$. And since $z = \frac{1}{2}u$, the three roots of the equation of the third degree are,

$$1,\ z = p = 1,$$

$$2,\ z = q = \frac{4 + \sqrt{15}}{2},$$

$$3,\ z = r = \frac{4 - \sqrt{15}}{2}.$$

783. Having therefore determined p, q, r, we have also their square roots; namely, $\sqrt{p} = 1$,

$$\sqrt{q} = \frac{\sqrt{(8 + 2\sqrt{15})}}{2}\text{*}, \text{ and } \sqrt{r} = \frac{\sqrt{(8 - 2\sqrt{15})}}{2}.$$

* This expression for the square root of q is obtained by multiplying the numerator and denominator of $\frac{4 + \sqrt{15}}{2}$ by 2, and extracting the root of the latter, in order to remove the surd: Thus, $\frac{4 + \sqrt{15}}{2} \times 2 = \frac{8 + 2\sqrt{15}}{4}$; and $\frac{\sqrt{(8 + 2\sqrt{15})}}{\sqrt{4}}$ $= \frac{\sqrt{(8 + 2\sqrt{15}}}{2}$.

But we have already seen, (Art. 675, and 676), that the square root of $a \pm \sqrt{b}$, when $\sqrt{(a^2 - b)} = c$, is expressed by $\sqrt{(a \pm \sqrt{b})} = \sqrt{(\frac{a+c}{2})} \pm \sqrt{(\frac{a-c}{2})}$: so that, as in the present case, $a = 8$, and $\sqrt{b} = 2\sqrt{15}$; and consequently, $b = 60$, and $c = \sqrt{(a^2 - b)} = 2$, we have $\sqrt{(8 + 2\sqrt{15})} = \sqrt{5} + \sqrt{3}$, and $\sqrt{(8 - 2\sqrt{15})} \ldots\ldots = \sqrt{5} - \sqrt{3}$. Hence, we have $\sqrt{p} = 1$, $\sqrt{q} = \frac{\sqrt{5} + \sqrt{3}}{2}$, and $\sqrt{r} = \frac{\sqrt{5} - \sqrt{3}}{2}$; wherefore, since we also know that the product of these quantities is positive, the four values of x will be:

1. $x = \sqrt{p} + \sqrt{q} + \sqrt{r} = 1 + \frac{\sqrt{5} + \sqrt{3} + \sqrt{5} - \sqrt{3}}{2} \ldots\ldots\ldots$
$= 1 + \sqrt{5}$,

2. $x = \sqrt{p} - \sqrt{q} - \sqrt{r} = 1 + \frac{-\sqrt{5} - \sqrt{3} - \sqrt{5} + \sqrt{3}}{2} \ldots\ldots$
$= 1 + \sqrt{5}$,

3. $x = -\sqrt{p} + \sqrt{q} - \sqrt{r} = -1 + \frac{\sqrt{5} + \sqrt{3} - \sqrt{5} + \sqrt{3}}{2} \ldots\ldots$
$= -1 + \sqrt{3}$,

4. $x = -1\sqrt{p} - \sqrt{q} + \sqrt{r} = -1 + \frac{-\sqrt{5} - \sqrt{3} + \sqrt{5} - \sqrt{3}}{2}$.
$= -1 - \sqrt{3}$.

Lastly, as we have $y = x + 2$, the four roots of the given equation are:

1. $y = 3 + \sqrt{5}$, 2. $y = 3 - \sqrt{5}$,
3. $y = 1 + \sqrt{3}$, 4. $y = 1 - \sqrt{3}$.

QUESTIONS FOR PRACTICE.

1. Given $z^4 - 4z^3 - 8z + 32 = 0$, to find the values of z. *Ans.* $4, 2, -1 + \sqrt{-3}, -1 - \sqrt{-3}$.

2. Given $y^4 - 4y^3 - 3y^2 - 4y + 1 = 0$, to find the values of y. *Ans.* $\frac{-1 \pm \sqrt{-3}}{2}$, and $\frac{5 \pm \sqrt{21}}{2}$.

3. Given $x^4 - 3x^2 - 4x = 3$, to find the values of x.

Ans. $\frac{1 \pm \sqrt{13}}{2}$, and $\frac{-1 \pm \sqrt{-3}}{2}$.