CHAP. XIII.

Of the Resolution *of* Equations *of the* Fourth Degree.

750. When the highest power of the quantity x rises to the fourth degree, we have *equations of the fourth degree*, the general form of which is

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

We shall, in the first place, consider *pure* equations of the fourth degree, the expression for which is simply $x^4 = f$; the root of which is immediately found by extracting the biquadrate root of both sides, since we obtain $x = \sqrt[4]{f}$.

751. As x^4 is the square of x^2, the calculation is greatly facilitated by beginning with the extraction of the square root; for we shall then have $x^2 = \sqrt{f}$; and, taking the square root again, we have $x = \sqrt[4]{f}$; so that $\sqrt[4]{f}$ is nothing but the square root of the square root of f.

For example, if we had the equation $x^4 = 2401$, we should immediately have $x^2 = 49$, and then $x = 7$.

752. It is true this is only one root; and since there are always three roots in an equation of the third degree, so also there are four roots in an equation of the fourth degree: but the method which we have explained will actually give those four roots. For, in the above example, we have not only $x^2 = 49$, but also $x^2 = -49$; now, the first value gives the two roots $x = 7$ and $x = -7$, and the second value gives $x = \sqrt{-49}$, $= 7\sqrt{-1}$, and $x = -\sqrt{-49} = -7\sqrt{-1}$; which are the four biquadrate roots of 2401. The same also is true with respect to other numbers.

753. Next to these pure equations, we shall consider others, in which the second and fourth terms are wanting, and which have the form $x^4 + fx^2 + g = 0$. These may be resolved by the rule for equations of the second degree; for if we make $x^2 = y$, we have $y^2 + fy + g = 0$, or $y^2 = -fy - g$, whence we deduce

$$y = -\tfrac{1}{2}f \pm \sqrt{(\tfrac{1}{4}f^2 - g)} = \left(\frac{-f \pm \sqrt{(f^2 - 4g)}}{2}\right).$$

Now, $x^2 = y$; so that $x = \pm\sqrt{\left(\frac{-f \pm \sqrt{(f^2 - 4g)}}{2}\right)}$, in which the double signs $\pm$ indicate all the four roots.

754. But whenever the equation contains all the terms, it may be considered as the product of four factors. In fact, if we multiply these four factors together, $(x-p) \times (x-q) \times (x-r) \times (x-s)$, we get the product $x^4 - (p+q+r+s)x^3 + (pq+pr+ps+qr+qs+rs)x^2 - (pqr+pqs+prs+qrs)x + pqrs$; and this quantity cannot be equal to 0, except when one of these four factors is $= 0$. Now, that may happen in four ways;

1. when $x = p$; 2. when $x = q$;
3. when $x = r$; 4. when $x = s$;

and consequently these are the four roots of the equation.

755. If we consider this formula with attention, we observe, in the second term, the sum of the four roots multiplied by $-x^3$; in the third term, the sum of all the possible products of two roots, multiplied by x^2; in the fourth term, the sum of the products of the roots combined three by three, multiplied by $-x$; lastly, in the fifth term, the product of all the four roots multiplied together.

756. As the last term contains the product of all the roots, it is evident that such an equation of the fourth degree can have no rational root, which is not a divisor of the last term. This principle, therefore, furnishes an easy method of determining all the rational roots, when there are any; since we have only to substitute successively for x all the divisors of the last term, till we find one which satisfies the terms of the equation: for having found such a root, for example, $x = p$, we have only to divide the equation by $x - p$, after having brought all the terms to one side, and then suppose the quotient $= 0$. We thus obtain an equation of the third degree, which may be resolved by the rules already given.

757. Now, for this purpose, it is absolutely necessary that all the terms should consist of integers, and that the first should have only unity for the coefficient; whenever, therefore, any terms contain fractions, we must begin by destroying those fractions, and this may always be done by substituting, instead of x, the quantity y, divided by a number which contains all the denominators of those fractions.

For example, if we have the equation

$$x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{3}x^2 - \tfrac{3}{4}x + \tfrac{1}{18} = 0,$$

as we find here fractions which have for denominators 2, 3, and multiples of these numbers, let us suppose $x = \dfrac{y}{6}$, and we shall then have

$$\frac{y^4}{6^4} - \frac{\frac{1}{2}y^3}{6^3} + \frac{\frac{1}{3}y^2}{6^2} - \frac{\frac{3}{4}y}{6} + \tfrac{1}{18} = 0,$$

an equation, which, multiplied by 6^4, becomes

$$y^4 - 3y^3 + 12y^2 - 162y + 72 = 0.$$

If we now wish to know whether this equation has rational roots, we must write, instead of y, the divisors of 72 successively, in order to see in what cases the formula would really be reduced to 0.

758. But as the roots may as well be positive as negative, we must make two trials with each divisor; one, supposing that divisor positive, the other, considering it as negative. However, the following rule will frequently enable us to dispense with this*. Whenever the signs + and − succeed each other regularly, the equation has as many positive roots as there are changes in the signs; and as many times as the same sign recurs without the other intervening, so many negative roots belong to the equation. Now, our example contains four changes of the signs, and no succession; so that all the roots are positive: and we have no need to take any of the divisors of the last term negatively.

759. Let there be given the equation

$$x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$$

We see here two changes of signs, and also two successions; whence we conclude, with certainty, that this equation contains two positive, and as many negative roots, which must all be divisors of the number 12. Now, its divisors being 1, 2, 3, 4, 6, 12, let us first try $x = +1$, which actually produces 0; therefore one of the roots is $x = 1$.

If we next make $x = -1$, we find $+1 - 2 - 7 + 8 + 12 = 21 - 9 = 12$: so that $x = -1$ is not one of the roots of the equation. Let us now make $x = 2$, and we again find the quantity $= 0$; consequently, another of the roots is $x = 2$; but $x = -2$, on the contrary, is found not to be a root. If we suppose $x = 3$, we have $81 + 54 - 63 - 24 + 12 = 60$, so that the supposition does not answer; but $x = -3$, giving $81 - 54 - 63 + 24 + 12 = 0$, this is evidently one of the roots sought. Lastly, when we try $x = -4$, we likewise see the equation reduced to nothing; so that all the four roots are rational, and have the following values: $x = 1$, $x = 2$, $x = -3$, and $x = -4$; and, ac-

* This rule is general for equations of all dimensions, provided there are no imaginary roots. The French ascribe it to Descartes, the English to Harriot; but the general demonstration of it was first given by M. l'Abbé de Gua. See the Memoires de l'Academie des Sciences de Paris, for 1741. F. T.

cording to the rule given above, two of these roots are positive, and the two others are negative.

760. But as no root could be determined by this method, when the roots are all irrational, it was necessary to devise other expedients for expressing the roots whenever this case occurs; and two different methods have been discovered for finding such roots, whatever be the nature of the equation of the fourth degree.

But before we explain those general methods, it will be proper to give the solution of some particular cases, which may frequently be applied with great advantage.

761. When the equation is such, that the coefficients of the terms succeed in the same manner, both in the direct and in the inverse order of the terms, as happens in the following equation *;

$$x^4 + mx^3 + nx^2 + mx + 1 = 0;$$

or in this other equation, which is more general:

$$x^4 + max^3 + na^2x^2 + ma^3x + a^4 = 0;$$

we may always consider such a formula as the product of two factors, which are of the second degree, and are easily resolved. In fact, if we represent this last equation by the product

$$(x^2 + pax + a^2) \times (x^2 + qax + a^2) = 0,$$

in which it is required to determine p and q in such a manner, that the above equation may be obtained, we shall find, by performing the multiplication,

$$x^4 + (p + q)ax^3 + (pq + 2)a^2x^2 + (p + q)a^3x + a^4 = 0;$$

and, in order that this equation may be the same as the former, we must have,

$$1.\ \ p + q = m,$$
$$2.\ \ pq + 2 = n,$$
and, consequently, $pq = n - 2$.

* These equations may be called *reciprocal*, for they are not at all changed by substituting $\frac{1}{x}$ for x. From this property it follows, that if a, for instance, be one of the roots, $\frac{1}{a}$ will be one likewise; for which reason such equations may be reduced to others of a dimension one half less. De Moivre has given, in his Miscellanea Analytica, page 71, general formulæ for the reduction of such equations, whatever be their dimension. F. T.

See also Wood's Algebra, the Complément des Elemens d'Algebra, by Lacroix, and Waring's Medit. Algeb. chap. 3.

Now, squaring the first of those equations, we have $p^2 + 2pq + q^2 = m^2$; and if from this we subtract the second, taken four times, or $4pq = 4n - 8$, there remains $p^2 - 2pq + q^2 = m^2 - 4n + 8$; and taking the square root, we find $p - q = \sqrt{(m^2 - 4n + 8)}$; also, $p + q = m$; we shall therefore have, by addition, $2p = m + \sqrt{(m^2 - 4n + 8)}$, or $p = \frac{m + \sqrt{(m^2 - 4n + 8)}}{2}$; and, by subtraction, $2q = m - \sqrt{(m^2 - 4n + 8)}$, or $q = \frac{m - \sqrt{(m^2 - 4n + 8)}}{2}$. Having therefore found p and q, we have only to suppose each factor $= 0$, in order to determine the value of x. The first gives $x^2 + pax + a^2 = 0$, or $x^2 = - pax - a^2$, whence we obtain $x = -\frac{pa}{2} \pm \sqrt{(\frac{p^2a^2}{4} - a^2)}$,

$$\text{or } x = -\frac{pa}{2} \pm \tfrac{1}{2}a\sqrt{(p^2 - 4)}.$$

The second factor gives $x = -\frac{qa}{2} \pm \frac{1}{2}a\sqrt{(q^2 - 4)}$;

and these are the four roots of the given equation.

762. To render this more clear, let there be given the equation $x^4 - 4x^3 - 3x^2 - 4x + 1 = 0$. We have here $a = 1$, $m = -4$, $n = -3$; consequently, $m^2 - 4n + 8 = 36$, and the square root of this quantity is $= 6$; therefore $p = \frac{-4+6}{2} = 1$, and $q = \frac{-4-6}{2} = -5$; whence result the four roots,

1st and 2d $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3} = \frac{-1 \pm \sqrt{(-3)}}{2}$; and

3d and 4th $x = \frac{5}{2} \pm \frac{1}{2}\sqrt{21} = \frac{5 \pm \sqrt{21}}{2}$; that is, the four roots of the given equation are:

$$1.\ x = \frac{-1 + \sqrt{-3}}{2}, \quad 2.\ x = \frac{-1 - \sqrt{-3}}{2},$$

$$3.\ x = \frac{5 + \sqrt{21}}{2}, \quad 4.\ x = \frac{5 - \sqrt{21}}{2}.$$

The first two of these roots are imaginary, or impossible; but the last two are possible; since we may express $\sqrt{21}$ to any degree of exactness, by means of decimal fractions. In fact, 21 being the same with 21·00000000, we have only to extract the square root, which gives $\sqrt{21} = 4{\cdot}5825$.

Since, therefore, $\sqrt{21} = 4{\cdot}5825$, the third root is very nearly $x = 4{\cdot}7912$, and the fourth, $x = 0{\cdot}2087$. It would have been easy to have determined these roots with still more precision: for we observe that the fourth root is very nearly $\frac{2}{10}$, or $\frac{1}{5}$, which value will answer the equation with sufficient exactness. In fact, if we make $x = \frac{1}{5}$, we find $\frac{1}{625} - \frac{4}{125} - \frac{3}{25} - \frac{4}{5} + 1 = \frac{31}{625}$. We ought however to have obtained 0, but the difference is evidently not great.

763. The second case in which such a resolution takes place, is the same as the first with regard to the coefficients, but differs from it in the signs, for we shall suppose that the second and the fourth terms have different signs; such, for example, as the equation

$$x^4 + max^3 + na^2x^2 - ma^3x + a^4 = 0,$$

which may be represented by the product,

$$(x^2 + pax - a^2) \times (x^2 + qax - a^2) = 0.$$

For the actual multiplication of these factors gives

$$x^4 + (p + q)ax^3 + (pq - 2)a^2x^2 - (p + q)a^3x + a^4,$$

a quantity equal to that which was given, if we suppose, in the first place, $p + q = m$, and in the second place, $pq - 2 = n$, or $pq = n + 2$; because in this manner the fourth terms become equal of themselves. If now we square the first equation, as before, (Art. 761.) we shall have $p^2 + 2pq + q^2 = m^2$; and if from this we subtract the second, taken four times, or $4pq = 4n + 8$, there will remain $p^2 - 2pq + q^2 = m^2 - 4n - 8$; the square root of which is $p - q = \sqrt{(m^2 - 4n - 8)}$, and thence, by adding $pn + q = m$, we obtain

$p = \dfrac{m + \sqrt{(m^2 - 4n - 8)}}{2}$; and, by subtracting $p + q$, . . .

$q = \dfrac{m - \sqrt{(m^2 - 4n - 8)}}{2}$. Having therefore found p and q,

we shall obtain from the first factor (as in Art. 761.) the two roots $x = -\frac{1}{2}pa \pm \frac{1}{2}a\sqrt{(p^2 + 4)}$, and from the second factor the two roots $x = -\frac{1}{2}qa \pm \frac{1}{2}a\sqrt{(q^2 + 4)}$, that is, we have the four roots of the equation proposed.

764. Let there be given the equation

$$x^4 - 3 \times 2x^3 + 3 \times 8x + 16 = 0.$$

Here we have $a = 2$, $m = -3$, and $n = 0$; so that $\sqrt{(m^2 - 4n - 8)} = 1, = p - q$; and, consequently,

$$p = \frac{-3+1}{2} = -1, \text{ and } q = \frac{-3-1}{2} = -2.$$

Therefore the first two roots are $x = 1 \pm \sqrt{5}$, and the last two are $x = 2 \pm \sqrt{8}$; so that the four roots sought will be,

$$1.\ x = 1 + \sqrt{5}, \qquad 2.\ x = 1 - \sqrt{5},$$
$$3.\ x = 2 + \sqrt{8}, \qquad 4.\ x = 2 - \sqrt{8}.$$

Consequently, the four factors of our equation will be $(x - 1 - \sqrt{5}) \times (x - 1 + \sqrt{5}) \times (x - 2 - \sqrt{8}) \times (x - 2 + \sqrt{8})$, and their actual multiplication produces the given equation; for the first two being multiplied together, give $x^2 - 2x - 4$, and the other two give $x^2 - 4x - 4$: now, these products multiplied together, make $x^4 - 6x^3 + 24x + 16$, which is the same equation that was proposed.

CHAP. XIV.

Of the Rule *of* Bombelli *for reducing the* Resolution *of* Equations *of the* Fourth Degree *to that of* Equations *of the* Third Degree.

765. We have already shewn how equations of the third degree are resolved by the rule of Cardan; so that the principal object, with regard to equations of the fourth degree, is to reduce them to equations of the third degree. For it is impossible to resolve, generally, equations of the fourth degree, without the aid of those of the third; since, when we have determined one of the roots, the others always depend on an equation of the third degree. And hence we may conclude, that the resolution of equations of higher dimensions presupposes the resolution of all equations of lower degrees.

766. It is now some centuries since Bombelli, an Italian, gave a rule for this purpose, which we shall explain in this chapter *.

Let there be given the general equation of the fourth degree, $x^4 + ax^3 + bx^2 + cx + d = 0$, in which the letters a, b, c, d, represent any possible numbers; and let us suppose that this equation is the same as $(x^2 + \frac{1}{2}ax + p)^2 - (qx + r)^2 = 0$, in which it is required to determine the letters p, q, and r, in order that we may obtain the equation

* This rule rather belongs to Louis Ferrari. It is improperly called the Rule of Bombelli, in the same manner as the rule discovered by Scipio Ferreo has been ascribed to Cardan. F. T.