the second, it is 8; and, according to the third, it is 10. The following statement shews, that all these will answer the conditions of the question:

Number of merchants - - - -	7	8	10
Each contributes $40x$ - - - -	280	320	400
In all they contribute $40x^2$ - - - The original stock was - - - -	1960 8240	2560 8240	4000 8240
The whole stock is $40x^2 + 8240$ -	10200	10800	12240
With this capital they gain as much per cent as there are partners - - - - - - -	714	864	1224
Each takes from it - - - - -	70	80	100
So that they all together take $10x^2$	490	640	1000
Therefore there remains - - -	224	224	224

CHAP. XII.

Of the Rule *of* Cardan, *or of* Scipio Ferreo.

734. When we have removed fractions from an equation of the third degree, according to the manner which has been explained, and none of the divisors of the last term are found to be a root of the equation, it is a certain proof, not only that the equation has no root in integer numbers, but also that a fractional root cannot exist; which may be proved as follows.

Let there be given the equation $x^3 - ax^2 + bx - c = 0$, in which, a, b, c, express integer numbers. If we suppose, for example, $x = \frac{3}{2}$, we shall have $\frac{27}{8} - \frac{9}{4}a + \frac{3}{2}b - c = 0$. Now, the first term only has 8 for the denominator; the others being either integer numbers, or numbers divided only by 4 or by 2, and therefore cannot make 0 with the first term. The same thing happens with every other fraction.

735. As in those fractions the roots of the equation are neither integer numbers, nor fractions, they are irrational, and, as it often happens, imaginary. The manner, therefore, of expressing them, and of determining the radical signs which affect them, forms a very important point, and deserves to be carefully explained. This method, called *Cardan's Rule*, is ascribed to *Cardan*, or more properly to *Scipio Ferreo*, both of whom lived some centuries since*.

736. In order to understand this rule, we must first attentively consider the nature of a cube, whose root is a binomial.

Let $a + b$ be that root; then the cube of it will be $a^3 + 3a^2b + 3ab^2 + b^3$, and we see that it is composed of the cubes of the two terms of the binomial, and beside that, of the two middle terms, $3a^2b + 3ab^2$, which have the common factor $3ab$, multiplying the other factor, $a + b$; that is to say, the two terms contain thrice the product of the two terms of the binomial, multiplied by the sum of those terms.

737. Let us now suppose $x = a + b$; taking the cube of each side, we have $x^3 = a^3 + b^3 + 3ab\,(a + b)$: and, since $a + b = x$, we shall have the equation, $x^3 = a^3 + b^3 + 3abx$, or $x^3 = 3abx + a^3 + b^3$, one of the roots of which we know to be $x = a + b$. Whenever, therefore, such an equation occurs, we may assign one of its roots.

For example, let $a = 2$ and $b = 3$; we shall then have the equation $x^3 = 18x + 35$, which we know with certainty to have $x = 5$ for one of its roots.

738. Farther, let us now suppose $a^3 = p$, and $b^3 = q$; we shall then have $a = \sqrt[3]{p}$ and $b = \sqrt[3]{q}$, consequently, $ab = \sqrt[3]{pq}$; therefore, whenever we meet with an equation, of the form $x^3 = 3x\sqrt[3]{pq} + p + q$, we know that one of the roots is $\sqrt[3]{p} + \sqrt[3]{q}$.

Now, we can determine p and q, in such a manner, that both $3\sqrt[3]{pq}$ and $p + q$ may be quantities equal to determinate numbers; so that we can always resolve an equation of the third degree, of the kind which we speak of.

739. Let, in general, the equation $x^3 = fx + g$ be proposed. Here, it will be necessary to compare f with $3\sqrt[3]{pq}$, and g with $p + q$; that is, we must determine p and q in

* This rule when first discovered by Scipio Ferreo was only for particular forms of cubics, but it was afterwards generalised by Tartalea and Cardan. See Montucla's Hist. Math.; also Dr. Hutton's Dictionary, article Algebra; and Professor Bonnycastle's Introduction to his Treatise on Algebra, Vol. 1. p. xii—xv.

such a manner, that $3\sqrt[3]{pq}$ may become equal to f, and $p+q=g$; for we then know that one of the roots of our equation will be $x=\sqrt[3]{p}+\sqrt[3]{q}$.

740. We have therefore to resolve these two equations,

$$3\sqrt[3]{pq}=f,$$
$$p+q=g.$$

The first gives $\sqrt[3]{pq}=\frac{f}{3}$; or $pq=\frac{f^3}{27}=\frac{1}{27}f^3$, and $4pq=\frac{4}{27}f^3$. The second equation, being squared, gives $p^2+2pq+q^2=g^2$; if we subtract from it $4pq=\frac{4}{27}f^3$, we have $q^2-2pq+q^2=g^2-\frac{4}{27}f^3$, and taking the square root of both sides, we have

$$p-q=\sqrt{(g^2-\tfrac{4}{27}f^3)}.$$

Now, since $p+q=g$, we have, by adding $p+q$ to one side of the equation, and its equal, g, to the other, $2p=g+\sqrt{(g^2-\frac{4}{27}f^3)}$; and, by subtracting $p-q$ from $p+q$, we have $2q=g-\sqrt{(g^2-\frac{4}{27}f^3)}$; consequently,

$$p=\frac{g+\sqrt{(g^2-\frac{4}{27}f^3)}}{2} \text{ and } q=\frac{g-\sqrt{(g^2-\frac{4}{27}f^3)}}{2}.$$

741. In a cubic equation, therefore, of the form $x^3=fx+g$, whatever be the numbers f and g, we have always for one of the roots

$$x=\sqrt[3]{\left(\frac{g+\sqrt{(g^2-\frac{4}{27}f^3)}}{2}\right)}+\sqrt[3]{\left(\frac{g-\sqrt{(g^2-\frac{4}{27}f^3)}}{2}\right)};$$

that is, an irrational quantity, containing not only the sign of the square root, but also the sign of the cube root; and this is the formula which is called *the Rule of Cardan.*

742. Let us apply it to some examples, in order that its use may be better understood.

Let $x^3=6x+9$. First, we shall have $f=6$, and $g=9$; so that $g^2=81$, $f^3=216$, $\frac{4}{27}f^3=32$; then $g^2-\frac{4}{27}f^3=49$, and $\sqrt{(g^2-\frac{4}{27}f^3)}=7$. Therefore, one of the roots of the given equation is

$$x=\sqrt[3]{\left(\frac{9+7}{2}\right)}+\sqrt[3]{\left(\frac{9-7}{2}\right)}=\sqrt[3]{\tfrac{16}{2}}+\sqrt[3]{\tfrac{2}{2}}=\sqrt[3]{8}+\sqrt[3]{1}= . .$$
$$2+1=3.$$

743. Let there be proposed the equation $x^3=3x+2$. Here, we shall have $f=3$ and $g=2$; and consequently, $g^2=4$, $f^3=27$, and $\frac{4}{27}f^3=4$; which gives $\sqrt{(g^2-\frac{4}{27}f^3)}=0$; whence it follows, that one of the roots is $x=\sqrt[3]{\left(\frac{2+0}{2}\right)}+\sqrt[3]{\left(\frac{2-0}{2}\right)}=1+1=2.$

744. It often happens, however, that, though such an equation has a rational root, that root cannot be found by the rule which we are now considering.

Let there be given the equation $x^3 = 6x + 40$, in which $x = 4$ is one of the roots. We have here $f = 6$ and $g = 40$; farther, $g^2 = 1600$, and $\frac{4}{27}f^3 = 32$; so that $g^2 - \frac{4}{27}f^3 = 1568$, and $\sqrt{(g^2 - \frac{4}{27}f^3)} = \sqrt{1568} = \ldots .$ $\sqrt{(4 \cdot 4 \cdot 49 \cdot 2)} = 28\sqrt{2}$; consequently one of the roots will be

$$x = \sqrt[3]{\left(\frac{40+28\sqrt{2}}{2}\right)} + \sqrt[3]{\left(\frac{40-28\sqrt{2}}{2}\right)} \text{ or}$$

$$x = \sqrt[3]{(20+14\sqrt{2})} + \sqrt[3]{(20-14\sqrt{2})};$$

which quantity is really $= 4$, although, upon inspection, we should not suppose it. In fact, the cube of $2 + \sqrt{2}$ being $20 + 14\sqrt{2}$, we have, reciprocally, the cube root of $20 + 14\sqrt{2}$ equal to $2 + \sqrt{2}$; in the same manner, $\sqrt[3]{(20 - 14\sqrt{2})} = 2 - \sqrt{2}$; wherefore our root $x = 2 + \sqrt{2} + 2 - \sqrt{2} = 4$*.

745. To this rule it might be objected, that it does not extend to all equations of the third degree, because the square of x does not occur in it; that is to say, the second term of the equation is wanting. But we may remark, that every complete equation may be transformed into another, in which the second term is wanting, which will therefore enable us to apply the rule.

To prove this, let us take the complete equation $x^3 - 6x^2 + 11x - 6 = 0$: where, if we take the third of the coefficient 6 of the second term, and make $x - 2 = y$, we shall have

$$x = y + 2, \qquad x^2 = y^2 + 4y + 4, \text{ and}$$
$$x^3 = y^3 + 6y^2 + 12y + 8;$$

$$\begin{aligned} \text{Consequently, } x^3 &= y^3 + 6y^2 + 12y + 8 \\ -6x^2 &= \quad\; -6y^2 - 24y - 24 \\ 11x &= \qquad\qquad\; 11y + 22 \\ -6 &= \qquad\qquad\qquad\; - 6 \end{aligned}$$

or, $x^3 - 6x^2 + 11x - 6 = y^3 - y$.

We have, therefore, the equation $y^3 - y = 0$, the resolu-

* We have no general rules for extracting the cube root of these binomials, as we have for the square root; those that have been given by various authors, all lead to a mixt equation of the third degree similar to the one proposed. However, when the extraction of the cube root is possible, the sum of the two radicals which represent the root of the equation, always becomes rational; so that we may find it immediately by the method explained, Art. 722. F. T.

tion of which it is evident, since we immediately perceive that it is the product of the factors

$$y(y^2 - 1) = y\,(y + 1) \times (y - 1) = 0.$$

If we now make each of these factors = 0, we have

$$1\begin{cases} y = 0, \\ x = 2, \end{cases} \qquad 2\begin{cases} y = -1, \\ x = \;\;\,1, \end{cases} \qquad 3\begin{cases} y = 1, \\ x = 3, \end{cases}$$

that is to say, the three roots which we have already found.

746. Let there now be given the general equation of the third degree, $x^3 + ax^2 + bx + c = 0$, of which it is required to destroy the second term.

For this purpose, we must add to x the third of the coefficient of the second term, preserving the same sign, and then write for this sum a new letter, as for example y, so that we shall have $x + \frac{1}{3}a = y$, and $x = y - \frac{1}{3}a$; whence results the following calculation:

$$x = y - \tfrac{1}{3}a,\ x^2 = y^2 - \tfrac{2}{3}ay + \tfrac{1}{9}a^2,$$
$$\text{and } x^3 = y^3 - ay^2 + \tfrac{1}{3}a^2y - \tfrac{1}{27}a^3;$$

Consequently,

$$\begin{aligned} x^3 &= y^3 - ay^2 + \tfrac{1}{3}a^2y - \tfrac{1}{27}a^3 \\ ax^2 &= \qquad\; ay^2 - \tfrac{2}{3}a^2y + \tfrac{1}{9}a^3 \\ bx &= \qquad\qquad\qquad by - \tfrac{1}{3}ab \\ c &= \qquad\qquad\qquad\qquad\quad c \end{aligned}$$

or, $y^3 - (\frac{1}{3}a - b)\,y + \frac{2}{27}a^3 - \frac{1}{3}ab + c = 0$, an equation in which the second term is wanting.

747. We are enabled, by means of this transformation, to find the roots of all equations of the third degree, as the following example will shew.

Let it be proposed to resolve the equation

$$x^3 - 6x^2 + 13x - 12 = 0.$$

Here it is first necessary to destroy the second term; for which purpose, let us make $x - 2 = y$, and then we shall have $x = y + 2$, $x^2 = y^2 + 4y + 4$, and $x^3 = y^3 + 6y^2 + 12y + 8$; therefore,

$$\begin{aligned} x^3 &= y^3 + 6y^2 + 12y + \;\,8 \\ -6x^2 &= \quad\;\; - 6y^2 - 24y - 24 \\ 13x &= \qquad\qquad\;\; 13y + 26 \\ -12 &= \qquad\qquad\qquad\;\; -12 \end{aligned}$$

which gives $y^3 + y - 2 = 0$; or $y^3 = -y + 2$.

And if we compare this equation with the formula, (Art. 741) $x^3 = fx + g$, we have $f = -1$, and $g = 2$; wherefore, $g^2 = 4$, and $\frac{4}{27}f^3 = -\frac{4}{27}$; also, $g^2 - \frac{4}{27}f^3 = 4 + \frac{4}{27} = \frac{112}{27}$, and $\sqrt{(g^2 - \frac{4}{27}f^3)} = \sqrt{\frac{112}{27}} = \dfrac{4\sqrt{21}}{9}$; consequently,

$$y = \left(\frac{2+\frac{4\sqrt{21}}{9}}{2}\right)^{\frac{1}{3}} + \left(\frac{2-\frac{4\sqrt{21}}{9}}{2}\right)^{\frac{1}{3}}, \text{ or}$$

$$y = \sqrt[3]{}(1 + \frac{2\sqrt{21}}{9}) + \sqrt[3]{}(1 - \frac{2\sqrt{21}}{9}), \text{ or}$$

$$y = \sqrt[3]{}(9 + \frac{2\sqrt{21}}{9}) + \sqrt[3]{}(9 - \frac{2\sqrt{21}}{9})$$

$$y = \sqrt{}(\frac{27+6\sqrt{21}}{27}) + \sqrt[3]{}(\frac{27-6\sqrt{21}}{27}) \text{ or}$$

$$y = \tfrac{1}{3}\sqrt[3]{}(27+6\sqrt{21})\,(+\tfrac{1}{3}\sqrt[3]{}(27-6\sqrt{21});$$

and it remains to substitute this value in $x = y + 2$.

748. In the solution of this example, we have been brought to a quantity doubly irrational; but we must not immediately conclude that the root is irrational: because the binomials $27 \pm 6\sqrt{21}$ might happen to be real cubes; and this is the case here; for the cube of

$\frac{3+\sqrt{21}}{2}$ being $\frac{216+48\sqrt{21}}{8} = 27+6\sqrt{21}$, it follows that

the cube root of $27 + 6\sqrt{21}$ is $\frac{3+\sqrt{21}}{2}$, and that the cube

root of $27 - 6\sqrt{21}$ is $\frac{3-\sqrt{21}}{2}$. Hence the value which we

found for y becomes

$$y = \tfrac{1}{3}(\frac{3+\sqrt{21}}{2}) + \tfrac{1}{3}(\frac{3-\sqrt{21}}{2}) = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

Now, since $y = 1$, we have $x = 3$ for one of the roots of the equation proposed, and the other two will be found by dividing the equation by $x - 3$.

$$\begin{array}{rl} x-3\,)\; x^3 - 6x^2 + 13x - 12\;(\,x^2 - 3x + 4 & \\ x^3 - 3x^2 \qquad\qquad\qquad & \\ \hline -3x^2 + 13x \qquad & \\ -3x^2 + 9x \qquad & \\ \hline 4x - 12 & \\ 4x - 12 & \\ \hline 0. & \end{array}$$

Also making the quotient $x^2 - 3x + 4 = 0$, we have $x^2 = 3x - 4$; and

$$x = \frac{3}{2} \pm \sqrt{(\frac{9}{4} - \frac{16}{4})} = \frac{3}{2} \pm \sqrt{-\frac{7}{4}} = \frac{3 \pm \sqrt{-7}}{2};$$

which are the other two roots, but they are imaginary.

749. It was, however, by chance, as we have remarked, that we were able, in the preceding example, to extract the cube root of the binomials that we obtained, which is the case only when the equation has a rational root; consequently, the rules of the preceding chapter are more easily employed for finding that root. But when there is no rational root, it is, on the other hand, impossible to express the root which we obtain in any other way, than according to the rule of Cardan; so that it is then impossible to apply reductions. For example, in the equation $x^3 = 6x + 4$, we have $f = 6$ and $g = 4$; so that $x = \sqrt[3]{(2 + 2\sqrt{-1})} + \sqrt[3]{(2 - 2\sqrt{-1})}$, which cannot be otherwise expressed*.

* In this example, we have $\frac{4}{27}f^3$ less than g^2, which is the well-known *irreducible case;* a case which is so much the more remarkable, as the three roots are then always real. We cannot here make use of Cardan's formula, except by applying the methods of approximation, such as transforming it into an infinite series. In the work spoken of in the Note, Art. 40, Lambert has given particular Tables, by which we may easily find the numerical values of the roots of cubic equations, in the irreducible, as well as the other cases. For this purpose we may also employ the ordinary Tables of sines. See the Spherical Astronomy of Mauduit, printed at Paris in 1765.

In the present work of EULER, we are not to look for all that might have been said on the direct and approximate resolutions of equations. He had too many curious and important objects, to dwell long upon this; but by consulting *l'Histoire des Mathematiques, l'Algebre de M. Clairaut, le Cours de Mathematiques de M. Bezout,* and the latter volumes of the Academical Memoirs of Paris and Berlin, the reader will obtain all that is known at present concerning the resolution of equations. F. T.

For a clear and explicit investigation of this method, the reader is also referred to Bonnycastle's Trigonometry; from which the following formulæ for the solution of the different cases of cubic equations are extracted.

1. $x^3 + px - q = 0.$

Put $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = \tan. z$, and $\sqrt[3]{(\tan. (45° - \frac{1}{2}z))} = \tan. u$;

Then $x = 2\sqrt{\frac{p}{3}} \times \cot. 2u$. Or, putting

Log. $\frac{q}{2} + 10 - \frac{3}{2}\log. \frac{p}{2} = \log. \tan. z$, and

QUESTIONS FOR PRACTICE.

1. Given $y^3 + 30y = 117$, to determine y. *Ans.* $y = 3$.
2. Given $y^3 - 36y = 91$, to find the value of y.

Ans. $y = 7$.

$$\tfrac{1}{3}(\log.\tan.(45^\circ - \tfrac{1}{2}z) + 20) = \log.\tan.u,$$

$$\text{Then } \log.x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cot.2u - 10.$$

$$2.\ x^3 + px + q = 0.$$

Put $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = \tan.z$, and $\sqrt[3]{(\tan.(45^\circ - \frac{1}{2}z))} = \tan.u$,

$$\text{Then } x = -2\sqrt{\frac{p}{3}} \times \cot.2u. \text{ Or, putting}$$

$$\text{Log.}\frac{q}{2} + 10 - \tfrac{3}{2}\log.\frac{p}{3} = \log.\tan.z, \text{ and}$$

$$\tfrac{1}{3}(\log.\tan.(45^\circ - \tfrac{1}{2}z) + 20) = \log.\tan.u,$$

$$\text{Then } \log.x = 10 - \tfrac{1}{2}\log.\frac{4p}{3} - \log.\cot.2u.$$

$$3.\ x^3 - px - q = 0.$$

This form has 2 cases, according as $\frac{2}{q}(\frac{p}{3})^{\frac{3}{2}}$ is less, or greater than 1.

$$\text{In the 1st case, put } \frac{2}{q}(\frac{p}{3})^{\frac{3}{2}} = \cos.z.$$

$$\text{And } \sqrt[3]{(\tan.45^\circ - \tfrac{1}{2}z)} = \tan.u;$$

$$\text{Then } x = 2\sqrt{\frac{p}{3}} \times \text{cosec}.2u. \text{ Or, putting}$$

$$10 + \tfrac{3}{2}\log.\frac{p}{3} - \log.\frac{q}{2} = \log.\cos.z, \text{ and}$$

$$\tfrac{1}{3}(\log.\tan.(45^\circ - \tfrac{1}{2}z) + 20) = \log.\tan.u;$$

$$\text{Then } \log.x = 10 + \log.\frac{4p}{3} - \log.\sin.2u.$$

In the 2d case, put $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = \cos.z$, and x will have the 3 following values:

$$x = +2\sqrt{\frac{p}{3}} \times \cos.\frac{z}{3}$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos.(60^\circ - \frac{z}{3})$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos.(60^\circ + \frac{z}{3}) \text{ or,}$$

3. Given $y^3 + 24y = 250$, to find the value of y.
Ans. $y = 5{\cdot}05$.

4. Given $y^6 - 3y^4 - 2y^2 - 8 = 0$, to find y. *Ans.* $y = 2$.

$$\text{Log. } x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.\frac{z}{3} - 10,$$

$$\text{Log. } x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.(60^\circ - \frac{z}{3}) - 10,$$

$$\text{Log. } x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.(60^\circ + \frac{z}{3}) - 10,$$

Taking the value of x, answering to log. x, positively in the first equation, and negatively in the two latter.

4. $x^3 - px + q = 0$.

This form, like the former, has also two cases, according as $\frac{2}{q}(\frac{p}{3})^{\frac{3}{2}}$ is less, or greater than 1.

In the 1st case, put $\frac{2}{q}(\frac{p}{3})^{\frac{3}{2}} = \cos. z$,

And $\sqrt[3]{(\tan.(45^\circ - \tfrac{1}{2}z))} = \tan. u$, as before;

Then $x = -2\sqrt{\frac{p}{3}}\ \text{cosec. } 2u$. Or, putting

$$10 + \tfrac{3}{2}\log.\frac{p}{3} - \log.\frac{q}{2} = \log.\cos. z, \text{ and}$$

$$\tfrac{1}{3}\{\log.(\tan. 45^\circ - \tfrac{1}{2}z) + 20\} = \log.\tan. u;$$

$$\text{Then, } -\log. x = 10 + \log\frac{4p}{3} - \log.\sin. 2u.$$

In the 2d case, put $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = \cos. z$, and x will have the 3 following values:

$$x = -2\sqrt{\frac{p}{3}} \times \cos.\frac{z}{3}$$

$$x = +2\sqrt{\frac{p}{3}} \times \cos.(60^\circ - \frac{z}{3})$$

$$x = +2\sqrt{\frac{p}{3}} \times \cos.(60^\circ + \frac{z}{3}). \text{ Or,}$$

$$\text{Log. } x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.\frac{z}{3} - 10,$$

$$\text{Log. } x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.(60^\circ - \frac{x}{3}) - 10,$$

$$\text{Log. } x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.(60^\circ + \frac{z}{3}) - 10,$$

5. Given $y^3 + 3y^2 + 9y = 13$, to determine y.
Ans. $y = 1$.

6. Given $x^3 - 6x = -9$, to find the value of x.
Ans. $x = -3$.

7. Given $x^3 - 6x^2 + 10x = 8$, to find x. *Ans.* $x = 4$.

8. Given $p^3 - \frac{193}{3}p = \frac{1150}{27}$, to find p. *Ans.* $p = 8\frac{1}{3}$.

9. Given $x^3 - \frac{13}{3}x = \frac{70}{27}$, to find x. *Ans.* $x = 2\frac{1}{3}$.

10. Given $a^2 - 91a = -330$, to find a. *Ans.* $a = 5$.

11. Given $y^3 - 19y = 30$, what is the value of y?
Ans. $y = 5$.

Taking the value of x, answering to log. x, negatively in the first equation, and positively in the two latter.

As an example of this mode of solution, in what is usually called the *Irreducible Case of Cubic Equations,* Let $x^3 - 3x = 1$, to find its 3 roots.

Here $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = \frac{1}{2}(\frac{3}{3})^{\frac{3}{2}} = \frac{1}{2} = .5 = \cos. 60° = z$, hence

$$x = 2\sqrt{\frac{p}{3}} \times \cos. \frac{z}{3} = 2 \cos. 20° = 1.8793852$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos. (60° - \frac{z}{3}) = -2 \cos. 40° = -1.5320888$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos. (60° + \frac{z}{3}) = -2 \cos. 80° = -0.3472964.$$

Also, let $x^3 - 3x = -1$, to find its three roots.

Here, as before, $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = .5 = \cos. 60° = z$, hence

$$x = -2\sqrt{\frac{p}{3}} \times \cos. \frac{z}{3} = -2 \cos. 20° = -1.8793852$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos. (60° - \frac{z}{3}) = 2 \cos. 40° = 1.5320888$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos. (60° + \frac{z}{3}) = 2 \cos. 80° = 0.3472964.$$

Where the roots are the negatives of those of the first case. For the mode of investigating these kinds of formulæ, see, in addition to the references already given, Cagnoli, Traité de Trigon. and Article Irreducible Case, in the Supplement to Dr. Hutton's Mathematical Dictionary.