CHAP. VIII.

Of Geometrical Proportions.

461. Two geometrical relations are equal when their ratios are equal; and this equality of two relations is called a *geometrical proportion*. Thus, for example, we write $a : b = c : d$, or $a : b :: c : d$, to indicate that the relation $a : b$ is equal to the relation $c : d$; but this is more simply expressed by saying a is to b as c to d. The following is such a proportion, $8 : 4 :: 12 : 6$; for the ratio of the relation $8 : 4$ is $\frac{2}{1}$, or 2, and this is also the ratio of the relation $12 : 6$.

462. So that $a : b :: c : d$ being a geometrical proportion, the ratio must be the same on both sides, consequently $\frac{a}{b} = \frac{c}{d}$; and, reciprocally, if the fractions $\frac{a}{b} = \frac{c}{d}$, we have $a : b :: c : d$.

463. A geometrical proportion consists therefore of four terms, such, that the first divided by the second gives the same quotient as the third divided by the fourth; and hence we deduce an important property, common to all geometrical proportions, which is, that the product of the first and the last term is always equal to the product of the second and third; or, more simply, that the product of the extremes is equal to the product of the means.

464. In order to demonstrate this property, let us take the geometrical proportion $a : b :: c : d$, so that $\frac{a}{b} = \frac{c}{d}$. Now, if we multiply both these fractions by b, we obtain $a = \frac{bc}{d}$, and multiplying both sides farther by d, we have $ad = bc$; but ad is the product of the extreme terms, and bc is that of the means, which two products are found to be equal.

465. Reciprocally, if the four numbers a, b, c, d, are such, that the product of the two extremes, a and d, is equal to the product of the two means, b and c, we are certain that they form a geometrical proportion: for, since $ad = bc$, we

have only to divide both sides by bd, which gives us $\frac{ad}{bd} = \frac{bc}{bd}$, or $\frac{a}{b} = \frac{c}{d}$, and consequently $a : b :: c : d$.

466. The four terms of a geometrical proportion, as $a : b :: c : d$, may be transposed in different ways, without destroying the proportion; for the rule being always, that the product of the extremes is equal to the product of the means, or $ad = bc$, we may say,

1st. $b : a :: d : c$; 2dly. $a : c :: b : d$;
3dly. $d : b :: c : a$; 4thly. $d : c :: b : a$.

467. Beside these four geometrical proportions, we may deduce some others from the same proportion, $a : b :: c : d$; for we may say, $a + b : a :: c + d : c$, or the first term, *plus* the second, is to the first, as the third, *plus* the fourth, is to the third; that is, $a + b : a :: c + d : c$.

We may farther say, the first, *minus* the second, is to the first, as the third, *minus* the fourth, is to the third, or $a - b : a :: c - d : c$. For, if we take the product of the extremes and the means, we have $ac - bc = ac - ad$, which evidently leads to the equality $ad = bc$.

And, in the same manner, we may demonstrate that $a + b : b :: c + d : d$; and that $a - b : b :: c - d : d$.

468. All the proportions which we have deduced from $a : b :: c : d$ may be represented generally as follows:

$$ma + nb : pa + qb :: mc + nd : pc + qd.$$

For the product of the extreme terms is $mpac + npbc + mqad + nqbd$; which, since $ad = bc$, becomes $mpac + npbc + mqbc + nqbd$; also the product of the mean terms is $mpac + mqbc + npad + nqbd$; or, since $ad = bc$, it is $mpac + mqbc + npbc + nqbd$; so that the two products are equal.

469. It is evident, therefore, that a geometrical proportion being given, for example, $6 : 3 :: 10 : 5$, an infinite number of others may be deduced from it. We shall, however, give only a few:

$3 : 6 :: 5 : 10$; $6 : 10 :: 3 : 5$; $9 : 6 :: 15 : 10$;
$3 : 3 :: 5 : 5$; $9 : 15 :: 3 : 5$; $9 : 3 :: 15 : 5$.

470. Since in every geometrical proportion the product of the extremes is equal to the product of the means, we may, when the three first terms are known, find the fourth from them. Thus, let the three first terms be $24 : 15 :: 40$ to the fourth term: here, as the product of the means is 600, the fourth term multiplied by the first, that is by 24, must

also make 600; consequently, by dividing 600 by 24 the quotient 25 will be the fourth term required, and the whole proportion will be 24 : 15 : : 40 : 25. In general, therefore, if the three first terms are $a : b :: c$; we put d for the unknown fourth letter; and since $ad = bc$, we divide both sides by a, and have $d = \frac{bc}{a}$; so that the fourth term is $\frac{bc}{a}$, which is found by multiplying the second term by the third, and dividing that product by the first.

471. This is the foundation of the celebrated *Rule of Three* in Arithmetic; for in that rule we suppose three numbers given, and seek a fourth, in geometrical proportion with those three; so that the first may be to the second, as the third is to the fourth.

472. But here it will be necessary to pay attention to some particular circumstances. First, if in two proportions the first and the third terms are the same, as in $a : b :: c : d$, and $a : f :: c : g$, then the two second and the two fourth terms will also be in geometrical proportion, so that $b : d :: f : g$; for the first proportion being transformed into this, $a : c :: b : d$, and the second into this, $a : c :: f : g$, it follows that the relations $b : d$ and $f : g$ are equal, since each of them is equal to the relation $a : c$. Thus, for example, if 5 : 100 : : 2 : 40, and 5 : 15 : : 2 : 6, we must have 100 : 40 : : 15 : 6.

473. But if the two proportions are such, that the mean terms are the same in both, I say that the first terms will be in an inverse proportion to the fourth terms: that is, if $a : b :: c : d$, and $f : b :: c : g$, it follows that $a : f :: g : d$. Let the proportions be, for example, 24 : 8 : : 9 : 3, and 6 : 8 : : 9 : 12, we have 24 : 6 : : 12 : 3; the reason is evident; for the first proportion gives $ad = bc$; and the second gives $fg = bc$; therefore $ad = fg$, and $a : f :: g : d$, or $a : g :: f : d$.

474. Two proportions being given, we may always produce a new one by separately multiplying the first term of the one by the first term of the other, the second by the second, and so on with respect to the other terms. Thus, the proportions $a : b :: c : d$, and $e : f :: g : h$ will furnish this, $ae : bf :: cg : dh$; for the first giving $ad = bc$, and the second giving $eh = fg$, we have also $adeh = bcfg$; but now $adeh$ is the product of the extremes, and $bcfg$ is the product of the means in the new proportion: so that the two products being equal, the proportion is true.

475. Let the two proportions be 6 : 4 : : 15 : 10 and 9 : 12 : : 15 : 20, their combination will give the proportion $6 \times 9 : 4 \times 12 :: 15 \times 15 : 10 \times 20$,

$$\text{or } 54 : 48 :: 225 : 200,$$
$$\text{or } \;\; 9 : \;\; 8 :: \;\; 9 : \;\; 8.$$

476. We shall observe, lastly, that if two products are equal, $ad = bc$, we may reciprocally convert this equality into a geometrical proportion; for we shall always have one of the factors of the first product in the same proportion to one of the factors of the second product, as the other factor of the second product is to the other factor of the first product: that is, in the present case, $a : c :: b : d$, or $a : b :: c : d$. Let $3 \times 8 = 4 \times 6$, and we may form from it this proportion, 8 : 4 : : 6 : 3, or this, 3 : 4 : : 6 : 8. Likewise, if $3 \times 5 = 1 \times 15$, we shall have 3 : 15 : : 1 : 5, or 5 . 1 : : 15 : 3, or 3 : 1 : : 15 : 5.

CHAP. IX.

Observations on the Rules *of* Proportion *and their* Utility.

477. This theory is so useful in the common occurrences of life, that scarcely any person can do without it. There is always a proportion between prices and commodities; and when different kinds of money are the subject of exchange, the whole consists in determining their mutual relations. The examples furnished by these reflections will be very proper for illustrating the principles of proportion, and shewing their utility by the application of them.

478. If we wished to know, for example, the relation between two kinds of money; suppose an old *louis d'or* and a *ducat:* we must first know the value of those pieces when compared with others of the same kind. Thus, an old louis being, at Berlin, worth 5 rixdollars and 8 drachms, and a ducat being worth 3 rixdollars, we may reduce these two values to one denomination; either to rixdollars, which gives the proportion 1L : 1D : : $5\frac{1}{3}$R : 3R, or : : 16 : 9; or to drachms, in which case we have 1L : 1D : : 128 : 72 : : 16 : 9; which proportions evidently give the true relation of the old louis to the ducat; for the equality of the products of the extremes and the means gives, in both cases, 9 louis