CHAP. II.

Of Arithmetical Proportion.

390. When two arithmetical ratios, or relations, are equal, this equality is called an *arithmetical proportion*.

Thus, when $a - b = d$ and $p - q = d$, so that the difference is the same between the numbers p and q as between the numbers a and b, we say that these four numbers form an arithmetical proportion; which we write thus, $a - b = p - q$, expressing clearly by this, that the difference between a and b is equal to the difference between p and q.

391. An arithmetical proportion consists therefore of four terms, which must be such, that if we subtract the second from the first, the remainder is the same as when we subtract the fourth from the third; thus, the four numbers 12, 7, 9, 4, form an arithmetical proportion, because $12 - 7 = 9 - 4$.

392. When we have an arithmetical proportion, as $a - b = p - q$, we may make the second and third terms change places, writing $a - p = b - q$: and this equality will be no less true; for, since $a - b = p - q$, add b to both sides, and we have $a = b + p - q$: then subtract p from both sides, and we have $a - p = b - q$.

In the same manner, as $12 - 7 = 9 - 4$, so also $12 - 9 = 7 - 4$*.

393. We may in every arithmetical proportion put the second term also in the place of the first, if we make the same transposition of the third and fourth; that is, if $a - b = p - q$, we have also $b - a = q - p$; for $b - a$ is the negative of $a - b$, and $q - p$ is also the negative of $p - q$; and thus, since $12 - 7 = 2 - 4$, we have also, $7 - 12 = 4 - 9$.

394. But the most interesting property of every arithmetical proportion is this, that the sum of the second and third term is always equal to the sum of the first and fourth. This property, which we must particularly consider, is expressed also by saying that the sum of the *means* is equal to the sum of the *extremes*. Thus, since $12 - 7 = 9 - 4$, we have $7 + 9 = 12 + 4$; the sum being in both cases 16.

* To indicate that those numbers form such a proportion, some authors write them thus: 12 . 7 : : 9 . 4.

395. In order to demonstrate this principal property, let $a - b = p - q$; then if we add to both $b + q$, we have $a + q = b + p$; that is, the sum of the first and fourth terms is equal to the sum of the second and third: and inversely, of four numbers, a, b, p, q, are such, that the sum of the second and third is equal to the sum of the first and fourth; that is, if $b + p = a + q$, we conclude, without a possibility of mistake, that those numbers are in arithmetical proportion, and that $a - b = p - q$; for, since $a + q = b + p$, if we subtract from both sides $b + q$, we obtain $a - b = p - q$.

Thus, the numbers 18, 13, 15, 10, being such, that the sum of the means ($13 + 15 = 28$) is equal to the sum of the extremes ($18 + 10 = 28$), it is certain that they also form an arithmetical proportion; and, consequently, that $18 - 13 = 15 - 10$.

396. It is easy, by means of this property, to resolve the following question. The first three terms of an arithmetical proportion being given, to find the fourth? Let a, b, p, be the first three terms, and let us express the fourth by q, which it is required to determine, then $a + q = b + p$; by subtracting a from both sides, we obtain $q = b + p - a$.

Thus, the fourth term is found by adding together the second and third, and subtracting the first from that sum. Suppose, for example, that 19, 28, 13, are the three first given terms, the sum of the second and third is 41; and taking from it the first, which is 19, there remains 22 for the fourth term sought, and the arithmetical proportion will be represented by $19 - 28 = 13 - 22$, or by $28 - 19 = 22 - 13$, or, lastly, by $28 - 22 = 19 - 13$.

397. When in arithmetical proportion the second term is equal to the third, we have only three numbers; the property of which is this, that the first, *minus* the second, is equal to the second, *minus* the third; or that the difference between the first and second number is equal to the difference between the second and third: the three numbers 19, 15, 11, are of this kind, since $19 - 15 = 15 - 11$.

398. Three such numbers are said to form a continued arithmetical proportion, which is sometimes written thus, $19 : 15 : 11$. Such proportions are also called *arithmetical progressions*, particularly if a greater number of terms follow each other according to the same law.

An arithmetical progression may be either *increasing*, or *decreasing*. The former distinction is applied when the terms go on increasing; that is to say, when the second exceeds the first, and the third exceeds the second by the same quantity; as in the numbers 4, 7, 10; and the decreasing progression is that in which the terms go on always diminishing by the same quantity, such as the numbers 9, 5, 1.

399. Let us suppose the numbers a, b, c, to be in arithmetical progression; then $a - b = b - c$, whence it follows, from the equality between the sum of the extremes and that of the means, that $2b = a + c$; and if we subtract a from both, we have $2b - a = c$.

400. So that when the first two terms a, b, of an arithmetical progression are given, the third is found by taking the first from twice the second. Let 1 and 3 be the first two terms of an arithmetical progression, the third will be $2 \times 3 - 1 = 5$; and these three numbers 1, 3, 5, give the proportion

$$1 - 3 = 3 - 5.$$

401. By following the same method, we may pursue the arithmetical progression as far as we please; we have only to find the fourth term by means of the second and third, in the same manner as we determined the third by means of the first and second, and so on. Let a be the first term, and b the second, the third will be $2b - a$, the fourth $4b - 2a - b = 3b - 2a$, the fifth $6b - 4a - 2b + a = 4b - 3a$, the sixth $8b - 6a - 3b + 2a = 5b - 4a$, the seventh $10b - 8a - 4b + 3a = 6b - 5a$, &c.

CHAP. III.

Of Arithmetical Progressions.

402. We have already remarked, that a series of numbers composed of any number of terms, which always increase, or decrease, by the same quantity, is called an *arithmetical progression.*

Thus, the natural numbers written in their order, as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. form an arithmetical progression, because they constantly increase by unity; and the series 25, 22, 19, 16, 13, 10, 7, 4, 1, &c. is also such a progression, since the numbers constantly decrease by 3.

403. The number, or quantity, by which the terms of an arithmetical progression become greater or less, is called the