CHAP. XII.

Of Infinite Decimal Fractions.

525. We have already seen, in logarithmic calculations, that Decimal Fractions are employed instead of Vulgar Fractions: the same are also advantageously employed in other calculations. It will therefore be very necessary to shew how a vulgar fraction may be transformed into a decimal fraction; and, conversely, how we may express the value of a decimal, by a vulgar fraction.

526. Let it be required, in general, to change the fraction $\frac{a}{b}$, into a decimal. As this fraction expresses the quotient of the division of the numerator a by the denominator b, let us write, instead of a, the quantity $a{\cdot}0000000$, whose value does not at all differ from that of a, since it contains neither tenth parts, hundredth parts, nor any other parts whatever. If we now divide the quantity by the number b, according to the common rules of division, observing to put the point in the proper place, which separates the decimal and the integers, we shall obtain the decimal sought. This is the whole of the operation, which we shall illustrate by some examples.

Let there be given first the fraction $\frac{1}{2}$, and the division in decimals will assume this form:

$$\frac{2)1{\cdot}0000000}{0{\cdot}5000000} = \tfrac{1}{2}.$$

Hence it appears, that $\frac{1}{2}$ is equal to $0{\cdot}5000000$ or to $0{\cdot}5$; which is sufficiently evident, since this decimal fraction represents $\frac{5}{10}$, which is equivalent to $\frac{1}{2}$.

527. Let now $\frac{1}{3}$ be the given fraction, and we shall have,

$$\frac{3)1{\cdot}0000000}{0{\cdot}3333333} = \tfrac{1}{3}.$$

This shews, that the decimal fraction, whose value is $\frac{1}{3}$, cannot, strictly, ever be discontinued, but that it goes on, ad infinitum, repeating always the number 3; which agrees with what has been already shewn, Art. 523; namely, that the fractions

$$\tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \tfrac{3}{10000}, \text{ \&c. } \textit{ad infinitum}, = \tfrac{1}{3}.$$

The decimal fraction which expresses the value of $\frac{2}{3}$, is also continued ad infinitum; for we have,

$$\frac{3)2\cdot0000000}{0\cdot6666666} = \frac{2}{3}.$$

Which is also evident from what we have just said, because $\frac{2}{3}$ is the double of $\frac{1}{3}$.

528. If $\frac{1}{4}$ be the fraction proposed, we have

$$\frac{4)1\cdot0000000}{0\cdot2500000} = \frac{1}{4}.$$

So that $\frac{1}{4}$ is equal to $0\cdot2500000$, or to $0\cdot25$: which is evidently true, since $\frac{2}{10}$, or $\frac{20}{100}$, $+ \frac{5}{100} = \frac{25}{100} = \frac{1}{4}$.

In like manner, we should have for the fraction $\frac{3}{4}$.

$$\frac{4)3\cdot0000000}{0\cdot7500000} = \frac{3}{4}.$$

So that $\frac{3}{4} = 0\cdot75$: and in fact

$$\frac{7}{10} + \frac{5}{100} = \frac{75}{100} = \frac{3}{4}.$$

The fraction $\frac{5}{4}$ is changed into a decimal fraction, by making

$$\frac{4)5\cdot0000000}{1\cdot2500000} = \frac{5}{4}.$$

Now, $1 + \frac{25}{100} = \frac{5}{4}$.

529. In the same manner, $\frac{1}{5}$ will be found equal to $0\cdot2$; $\frac{2}{5} = 0\cdot4$; $\frac{3}{5} = 0\cdot6$; $\frac{4}{5} = 0\cdot8$; $\frac{5}{5} = 1$; $\frac{6}{5} = 1\cdot2$, &c.

When the denominator is 6, we find $\frac{1}{6} = 0\cdot1666666$, &c. which is equal to $0\cdot666666 - 0\cdot5$: but $0\cdot666666 = \frac{2}{3}$, and $0\cdot5 = \frac{1}{2}$, wherefore $0\cdot1666666 = \frac{2}{3} - \frac{1}{2}$; or $\frac{4}{6} - \frac{3}{6} = \frac{1}{6}$.

We find, also, $\frac{2}{6} = 0\cdot333333$, &c. $= \frac{1}{3}$; but $\frac{3}{6}$ becomes $0\cdot5000000 = \frac{1}{2}$; also, $\frac{5}{6} = 0\cdot833333 = 0\cdot333333 + 0\cdot5$, that is to say, $\frac{1}{3} + \frac{1}{2}$; or $\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$.

530. When the denominator is 7, the decimal fractions become more complicated. For example, we find $\frac{1}{7} = 0\cdot142857$; however it must be observed that these six figures are continually repeated. To be convinced, therefore, that this decimal fraction precisely expresses the value of $\frac{1}{7}$, we may transform it into a geometrical progression, whose first term is $\frac{142857}{1000000}$, the ratio being $\frac{1}{1000000}$; and consequently, the sum $= \dfrac{\frac{142857}{1000000}}{1 - \frac{1}{1000000}} = \frac{142857}{999999} = \frac{1}{7}$.

531. We may prove, in a manner still more easy, that the decimal fraction, which we have found, is exactly equal to $\frac{1}{7}$; for, by substituting for its value the letter s, we have

$$\begin{aligned}
s &= 0{\cdot}142857142857142857, \&c \\
10s &= 1{\cdot}\ 42857142857142857, \&c. \\
100s &= 14{\cdot}\ 2857142857142857, \&c. \\
1000s &= 142{\cdot}\ 857142857142857, \&c. \\
10000s &= 1428{\cdot}\ 57142857142857, \&c. \\
100000s &= 14285{\cdot}\ 7142857142857, \&c. \\
1000000s &= 142857{\cdot}\ 142857142857, \&c. \\
\text{Subtract } s &= \qquad 0{\cdot}\ 142857142857, \&c. \\
\hline
999999s &= 142857.
\end{aligned}$$

And, dividing by 999999, we have $s = \frac{142857}{999999} = \frac{1}{7}$. Wherefore the decimal fraction, which was represented by s, is $= \frac{1}{7}$.

532. In the same manner, $\frac{2}{7}$ may be transformed into a decimal fraction, which will be 0·28571428, &c. and this enables us to find more easily the value of the decimal fraction which we have represented by s; because 0·28571428, &c. must be the double of it, and, consequently, $= 2s$. Now we have seen that

$$\begin{aligned}
100s &= 14{\cdot}28571428571, \&c. \\
\text{So that subtracting}\quad 2s &= 0{\cdot}28571428571, \&c. \\
\hline
\text{there remains } 98s &= 14 \\
\text{wherefore}\quad s &= \tfrac{14}{98} = \tfrac{1}{7}.
\end{aligned}$$

We also find $\frac{3}{7} = 0{\cdot}42857142857$, &c. which, according to our supposition, must be equal to $3s$; and we have found that

$$\begin{aligned}
10s &= 1{\cdot}42857142857, \&c. \\
\text{So that subtracting}\quad 3s &= 0{\cdot}42857142857, \&c. \\
\hline
\text{we have}\quad 7s &= 1, \text{ wherefore } s = \tfrac{1}{7}.
\end{aligned}$$

533. When a proposed fraction, therefore, has the denominator 7, the decimal fraction is infinite, and 6 figures are continually repeated; the reason of which is easy to perceive, namely, that when we continue the division, a remainder must return, sooner or later, which we have had already. Now, in this division, 6 different numbers only can form the remainder, namely 1, 2, 3, 4, 5, 6; so that, at least, after the sixth division, the same figures must return; but when the denominator is such as to lead to a division without remainder, these cases do not happen.

534. Suppose now that 8 is the denominator of the fraction proposed: we shall find the following decimal fractions:

$\frac{1}{8} = 0{\cdot}125$; $\frac{2}{8} = 0{\cdot}25$; $\frac{3}{8} = 0{\cdot}375$; $\frac{4}{8} = 0{\cdot}5$;
$\frac{5}{8} = 0{\cdot}625$; $\frac{6}{8} = 0{\cdot}75$; $\frac{7}{8} = 0{\cdot}875$, &c.

535. If the denominator be 9, we have

$\frac{1}{9} = 0{\cdot}111$, &c. $\frac{2}{9} = 0{\cdot}222$, &c. $\frac{3}{9} = 0{\cdot}333$, &c.

And if the denominator be 10, we have $\frac{1}{10} = 0{\cdot}1$, $\frac{2}{10} = 0{\cdot}2$, $\frac{3}{10} = 0{\cdot}3$. This is evident from the nature of decimals, as also that $\frac{1}{100} = 0{\cdot}01$; $\frac{37}{100} = 0{\cdot}37$; $\frac{256}{1000} = 0{\cdot}256$; $\frac{24}{10000} = 0{\cdot}0024$, &c.

536. If 11 be the denominator of the given fraction, we shall have $\frac{1}{11} = 0{\cdot}0909090$, &c. Now, suppose it were required to find the value of this decimal fraction: let us call it s, and we shall have

$$s = 0{\cdot}090909,$$
$$10s = 00{\cdot}909090,$$
$$100s = 9{\cdot}09090.$$

If, therefore, we subtract from the last the value of s, we shall have $99s = 9$, and consequently $s = \frac{9}{99} = \frac{1}{11}$: thus, also,

$$\tfrac{2}{11} = 0{\cdot}181818, \text{ \&c.}$$
$$\tfrac{3}{11} = 0{\cdot}272727, \text{ \&c.}$$
$$\tfrac{6}{11} = 0{\cdot}545454, \text{ \&c.}$$

537. There are a great number of decimal fractions, therefore, in which one, two, or more figures constantly recur, and which continue thus to infinity. Such fractions are curious, and we shall shew how their values may be easily found *.

* These recurring decimals furnish many interesting researches; I had entered upon them, before I saw the present *Algebra,* and should perhaps have prosecuted my inquiry, had I not likewise found a Memoir in the *Philosophical Transactions* for 1769, entitled *The Theory of circulating Fractions.* I shall content myself with stating here the reasoning with which I began.

Let $\frac{n}{d}$ be any real fraction irreducible to lower terms. And suppose it were required to find how many decimal places we must reduce it to, before the same terms will return again. In order to determine this, I begin by supposing that $10n$ is greater than d; if that were not the case, and only $100n$ or $1000n > d$, it would be necessary to begin with trying to reduce $\frac{10n}{d}$ or $\frac{100n}{d}$, &c. to less terms, or to a fraction $\frac{n^1}{d^1}$.

This being established, I say that the same period can return only when the same remainder n returns in the continual division.

Let us first suppose, that a single figure is constantly repeated, and let us represent it by a, so that $s = 0{\cdot}aaaaaaa$. We have

$$10s = a{\cdot}aaaaaaa$$
and subtracting $s = 0{\cdot}aaaaaaa$

we have $9s = a$; wherefore $s = \frac{a}{9}$.

538. When two figures are repeated, as ab, we have $s = 0{\cdot}ababab$. Therefore $100s = ab{\cdot}ababab$; and if we subtract s from it, there remains $99s = ab$; consequently, $s = \frac{ab}{99}$.

When three figures, as abc, are found repeated, we have $s = 0{\cdot}abcabcabc$; consequently, $1000s = abc{\cdot}abcabc$; and subtracting s from it, there remains $999s = abc$; wherefore $s = \frac{abc}{999}$, and so on.

Whenever, therefore, a decimal fraction of this kind oc-

Suppose that when this happens we have added s cyphers, and that q is the integral part of the quotient; then abstracting from the point, we shall have $\frac{n \times 10^s}{d} = q + \frac{n}{d}$; wherefore $q = \frac{n}{d} \times (10^s - 1)$. Now, as q must be an integer number, it is required to determine the least integer number for s, such that $\frac{n}{d} \times (10^s - 1$, or only that $\frac{10^s - 1}{d}$, may be an integer number.

This problem requires several cases to be distinguished: the first is that in which d is a divisor of 10, or of 100, or of 1000, &c. and it is evident that in this case there can be no circulating fraction. For the second case, we shall take that in which d is an odd number, and not a factor of any power of 10; in this case, the value of s may rise to $d - 1$, but frequently it is less. A third case is that in which d is even, and, consequently, without being a factor of any power of 10, has nevertheless a common divisor with one of those powers: this common divisor can only be a number of the form 2^c; so that if, $\frac{d}{2^c} = e$, I say, the period will be the same as for the fraction $\frac{n}{d}$, but will not commence before the figure represented by c. This case comes to the same therefore with the second case, on which it is evident the theory depends. F. T.

curs, it is easy to find its value. Let there be given, for example, 0·296296: its value will be $\frac{296}{999} = \frac{8}{27}$, by dividing both its terms by 37.

This fraction ought to give again the decimal fraction proposed; and we may easily be convinced that this is the real result, by dividing 8 by 9, and then that quotient by 3, because $27 = 3 \times 9$: thus, we have

$$\begin{array}{r} 9)\ 8\cdot000000 \\ \hline 3)\ 0\cdot888888 \\ \hline 0\cdot296296,\ \&c. \end{array}$$

which is the decimal fraction that was proposed.

539. Suppose it were required to reduce the fraction $\frac{1}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}$, to a decimal. The operation would be as follows:

$$\begin{array}{r} 2)\ 1\cdot00000000000000 \\ \hline 3)\ 0\cdot50000000000000 \\ \hline 4)\ 0\cdot16666666666666 \\ \hline 5)\ 0\cdot04166666666666 \\ \hline 6)\ 0\cdot00833333333333 \\ \hline 7)\ 0\cdot00138888888888 \\ \hline 8)\ 0\cdot00019841269841 \\ \hline 9)\ 0\cdot00002480158730 \\ \hline 10)\ 0\cdot00000275573192 \\ \hline 0\cdot00000027557319 \end{array}$$