compounded of the number of workmen and that of the days which they have been employed.

If, for example, 25 sous per day be given to one mason, and it is required what must be paid to 24 masons who have worked for 50 days, we state the calculation thus:

$$\begin{array}{l} 1 : 24 \\ 1 : 50 \\ \hline 1 : 1200 :: 25 : 30000 \text{ sous, or 1500 francs.} \end{array}$$

In these examples, five things being given, the rule which serves to resolve them is called, in books of arithmetic, The Rule of Five, or Double Rule of Three.

CHAP. XI.

Of Geometrical Progressions.

505. A series of numbers, which are always becoming a certain number of times greater, or less, is called a *geometrical progression*, because each term is constantly to the following one in the same geometrical ratio: and the number which expresses how many times each term is greater than the preceding, is called the *exponent*, or *ratio*. Thus, when the first term is 1 and the exponent, or ratio, is 2, the geometrical progression becomes,

Terms	1	2	3	4	5	6	7	8	9	&c.
Prog.	1,	2,	4,	8,	16,	32,	64,	128,	256,	&c.

The numbers 1, 2, 3, &c. always marking the place which each term holds in the progression.

506. If we suppose, in general, the first term to be a, and the ratio b, we have the following geometrical progression:

$$\begin{array}{l} 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8 \ldots\ldots n. \\ \textit{Prog. } a,\ ab,\ ab^2,\ ab^3,\ ab^4,\ ab^5,\ ab^6,\ ab^7 \ldots\ldots ab^{n-1}. \end{array}$$

So that, when this progression consists of n terms, the last term is ab^{n-1}. We must, however, remark here, that if the ratio b be greater than unity, the terms increase continually; if $b = 1$, the terms are all equal; lastly, if b be less than 1, or a fraction, the terms continually decrease. Thus, when $a = 1$, and $b = \frac{1}{2}$, we have this geometrical progression:

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{64}, \tfrac{1}{128}, \&c.$$

507. Here therefore we have to consider:

1. The first term, which we have called a.
2. The exponent, which we call b.
3. The number of terms, which we have expressed by n.
4. And the last term, which, we have already seen, is expressed by ab^{n-1}.

So that, when the first three of these are given, the last term is found by multiplying the $n-1$ power of b, or b^{n-1}, by the first term a.

If, therefore, the 50th term of the geometrical progression 1, 2, 4, 8, &c. were required, we should have $a = 1$, $b = 2$, and $n = 50$; consequently the 50th term would be 2^{49}; and as $2^9 = 512$, we shall have $2^{10} = 1024$; wherefore the square of 2^{10}, or 2^{20}, $= 1048576$, and the square of this number, which is 1099511627776, $= 2^{40}$. Multiplying therefore this value of 2^{40} by 2^9, or 512, we have $2^{49} = 562949953421312$ for the 50th term.

508. One of the principal questions which occurs on this subject, is to find the sum of all the terms of a geometrical progression; we shall therefore explain the method of doing this. Let there be given, first, the following progression, consisting of ten terms:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512,$$

the sum of which we shall represent by s, so that

$$s=1+2+4+8+16+32+64+128+256+512;$$

doubling both sides, we shall have

$$2s=2+4+8+16+32+64+128+256+512+1024;$$

and subtracting from this the progression represented by s, there remains $s = 1024 - 1 = 1023$; wherefore the sum required is 1023.

509. Suppose now, in the same progression, that the number of terms is undetermined, that is, let them be generally represented by n, so that the sum in question, or

$$s, =1+2+2^2+2^3+2^4 \ldots\ldots 2^{n-1}$$

If we multiply by 2, we have

$$2s=2+2^2+2^3+2^4 \ldots\ldots\ldots 2^n,$$

then subtracting from this equation the preceding one, we have $s = 2^n - 1$. It is evident, therefore, that the sum required is found, by multiplying the last term, 2^{n-1}, by the exponent 2, in order to have 2^n, and subtracting unity from that product,

510. This is made still more evident by the following

examples, in which we substitute successively for n, the numbers 1, 2, 3, 4, &c.

$1=1$; $1+2=3$; $1+2+4=7$; $1+2+4+8=15$; $1+2+4+8+16=31$; $1+2+4+8+16+32=63$, &c.

511. On this subject, the following question is generally proposed. A man offers to sell his horse on the following condition; that is, he demands 1 penny for the first nail, 2 for the second, 4 for the third, 8 for the fourth, and so on, doubling the price of each succeeding nail. It is required to find the price of the horse, the nails being 32 in number?

This question is evidently reduced to finding the sum of all the terms of the geometrical progression 1, 2, 4, 8, 16, &c. continued to the 32d term. Now, that last term is 2^{31}; and, as we have already found $2^{20}=1048576$, and $2^{10}=1024$, we shall have $2^{20}\times 2^{10}=2^{30}=1073741824$; and multiplying again by 2, the last term $2^{31}=2147483648$; doubling therefore this number, and subtracting unity from the product, the sum required becomes 4294967295 pence; which being reduced, we have 17895697*l.* 1*s.* 3*d.* for the price of the horse.

512. Let the ratio now be 3, and let it be required to find the sum of the geometrical progression 1, 3, 9, 27, 81, 243, 729, consisting of 7 terms.

Calling the sum s as before, we have

$$s=1+3+9+27+81+243+729.$$

And multiplying by 3,

$$3s=3+9+27+81+243+729+2187.$$

Then subtracting the former series from the latter, we have $2s=2187-1=2186$: so that the double of the sum is 2186, and consequently the sum required is 1093.

513. In the same progression, let the number of terms be n, and the sum s; so that

$$s=1+3+3^2+3^3+3^4+\ldots.3^{n-1}.$$

If now we multiply by 3, we have

$$3s=3+3^2+3^3+3^4+\ldots.3^n.$$

Then subtracting from this expression the value of s, as before, we shall have $2s=3^n-1$; therefore $s=\frac{3^n-1}{2}$. So that the sum required is found by multiplying the last term by 3, subtracting 1 from the product, and dividing the remainder by 2; as will appear, also, from the following particular cases:

$$1 \quad - \quad - \quad - \quad - \quad \frac{(1\times 3)-1}{2} = 1$$

$$1+3 \quad - \quad - \quad - \quad \frac{(3\times 3)-1}{2} = 4$$

$$1+3+9 \quad - \quad - \quad \frac{(3\times 9)-1}{2} = 13$$

$$1+3+9+27 \quad - \quad \frac{(3\times 27)-1}{2} = 40$$

$$1+3+9+27+81 \quad \frac{(3\times 81)-1}{2} = 121.$$

514. Let us now suppose, generally, the first term to be a, the ratio b, the number of terms n, and their sum s, so that

$$s = a + ab + ab^2 + ab^3 + ab^4 + \dots ab^{n-1}.$$

If we multiply by b, we have

$$bs = ab + ab^2 + ab^3 + ab^4 + ab^5 + \dots ab^n,$$

and taking the difference between this and the above equation, there remains $(b-1)\,s = ab^n - a$; whence we easily deduce the sum required $s = \frac{a.(b^n-1)}{b-1}$. Consequently, the sum of any geometrical progression is found, by multiplying the last term by the ratio, or exponent of the progression, and dividing the difference between this product and the first term, by the difference between 1 and the ratio.

515. Let there be a geometrical progression of seven terms, of which the first is 3; and let the ratio be 2: we shall then have $a = 3$, $b = 2$, and $n = 7$; therefore the last term is 3×2^6, or 3×64, $= 192$; and the whole progression will be

$$3, 6, 12, 24, 48, 96, 192.$$

Farther, if we multiply the last term 192 by the ratio 2, we have 384; subtracting the first term, there remains 381; and dividing this by $b - 1$, or by 1, we have 381 for the sum of the whole progression.

516. Again, let there be a geometrical progression of six terms, of which the first is 4; and let the ratio be $\frac{3}{2}$: then the progression is

$$4, 6, 9, \tfrac{27}{2}, \tfrac{81}{4}, \tfrac{243}{8}.$$

If we multiply the last term by the ratio, we shall have $\frac{729}{16}$; and subtracting the first term $= \frac{64}{16}$, the remainder is $\frac{665}{16}$; which, divided by $b - 1 = \frac{1}{2}$, gives $\frac{665}{8} = 83\frac{1}{8}$ for the sum of the series.

517. When the exponent is less than 1, and, consequently, when the terms of the progression continually diminish, the sum of such a decreasing progression, carried on to infinity, may be accurately expressed.

For example, let the first term be 1, the ratio $\frac{1}{2}$, and the sum s, so that:

$$s = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64}, \text{ \&c.}$$

ad infinitum.

If we multiply by 2, we have

$$2s = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} +, \text{ \&c.}$$

ad infinitum: and, subtracting the preceding progression, there remains $s = 2$ for the sum of the proposed infinite progression.

518. If the first term be 1, the ratio $\frac{1}{3}$, and the sum s; so that

$$s = 1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} +, \text{ \&c. ad infinitum:}$$

Then multiplying the whole by 3, we have

$$3s = 3 + 1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} +, \text{ \&c. ad infinitum;}$$

and subtracting the value of s, there remains $2s = 3$; wherefore the sum $s = 1\frac{1}{2}$.

519. Let there be a progression whose sum is s, the first term 2, and the ratio $\frac{3}{4}$; so that

$$s = 2 + \tfrac{3}{2} + \tfrac{9}{8} + \tfrac{27}{32} + \tfrac{81}{128} +, \text{ \&c. ad infinitum.}$$

Multiplying by $\frac{4}{3}$, we have

$$\tfrac{4}{3}s = \tfrac{8}{3} + 2 + \tfrac{3}{2} + \tfrac{9}{8} + \tfrac{27}{32} + \tfrac{81}{128} +, \text{ \&c. ad infinitum;}$$

and subtracting from this progression s, there remains $\frac{1}{3}s = \frac{8}{3}$: wherefore the sum required is 8.

520. If we suppose, in general, the first term to be a, and the ratio of the progression to be $\frac{b}{c}$, so that this fraction may be less than 1, and consequently c greater than b; the sum of the progression, carried on ad infinitum, will be found thus:

$$\text{Make } s = a + \frac{ab}{c} + \frac{ab^2}{c^2} + \frac{ab^3}{c^3} + \frac{ab^4}{c^4} +, \text{ \&c.}$$

Then multiplying by $\frac{b}{c}$, we shall have

$$\frac{b}{c}s = \frac{ab}{c} + \frac{ab^2}{c^2} + \frac{ab^3}{c^3} + \frac{ab^4}{c^4} +, \text{ \&c. ad infinitum;}$$

and subtracting this equation from the preceding, there remains $(1 - \frac{b}{c})\, s = a$.

Consequently, $s = \dfrac{a}{1-\dfrac{b}{c}} = \dfrac{ac}{c-b}$, by multiplying both the numerator and denominator by c.

The sum of the infinite geometrical progression proposed is, therefore, found by dividing the first term a by 1 minus the ratio, or by multiplying the first term a by the denominator of the ratio, and dividing the product by the same denominator diminished by the numerator of the ratio *.

521. In the same manner we find the sums of progressions, the terms of which are alternately affected by the signs + and —. Suppose, for example,

$$s = a - \frac{ab}{c} + \frac{ab^2}{c^2} - \frac{ab^3}{c^3} + \frac{ab^4}{c^4} -, \text{ \&c.}$$

Multiplying by $\dfrac{b}{c}$, we have,

$$\frac{b}{c}s = \frac{ab}{c} - \frac{ab^2}{c^2} + \frac{ab^3}{c^3} - \frac{ab^4}{c^4}, \text{ \&c.}$$

And, adding this equation to the preceding, we obtain $(1 + \dfrac{b}{c})s = a$: whence we deduce the sum required, $s = \dfrac{a}{1+\dfrac{b}{c}}$, or $s = \dfrac{ac}{c+b}$.

522. It is evident, therefore, that if the first term $a = \frac{3}{5}$, and the ratio be $\frac{2}{5}$, that is to say, $b = 2$, and $c = 5$, we shall find the sum of the progression $\frac{3}{5} + \frac{6}{25} + \frac{12}{125} + \frac{24}{625} +$, &c. $= 1$; since, by subtracting the ratio from 1, there remains $\frac{3}{5}$, and by dividing the first term by that remainder, the quotient is 1.

It is also evident, if the terms be alternately positive and negative, and the progression assume this form:

$$\tfrac{3}{5} - \tfrac{6}{25} + \tfrac{12}{125} - \tfrac{24}{625} +, \text{ \&c.}$$

that the sum will be

$$\frac{a}{1+\dfrac{b}{c}} = \frac{\frac{3}{5}}{\frac{7}{5}} = \tfrac{3}{7}.$$

523. Again: let there be proposed the infinite progression,

* This particular case is included in the general Rule, Art. 514.

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} +, \&c.$$

The first term is here $\frac{3}{10}$, and the ratio is $\frac{1}{10}$; therefore subtracting this last from 1, there remains $\frac{9}{10}$, and, if we divide the first term by this fraction, we have $\frac{1}{3}$ for the sum of the given progression. So that taking only one term of the progression, namely, $\frac{3}{10}$, the error would be $\frac{1}{30}$.

And taking two terms, $\frac{3}{10} + \frac{3}{100} = \frac{33}{100}$, there would still be wanting $\frac{1}{300}$ to make the sum, which we have seen is $\frac{1}{3}$.

524. Let there now be given the infinite progression,

$$9 + \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} +, \&c.$$

The first term is 9, and the ratio is $\frac{1}{10}$. So that 1 minus the ratio is $\frac{9}{10}$; and $\frac{9}{\frac{9}{10}} = 10$, the sum required: which series is expressed by a decimal fraction, thus, 9·9999999, &c.

QUESTIONS FOR PRACTICE.

1. A servant agreed with a master to serve him eleven years without any other reward for his service than the produce of one grain of wheat for the first year; and that product to be sown the second year, and so on from year to year till the end of the time, allowing the increase to be only in a tenfold proportion. What was the sum of the whole produce?

Ans. 11111111110 grains.

N. B. It is farther required, to reduce this number of grains to the proper measures of capacity, and then by supposing an average price of wheat to compute the value of the corns in money.

2. A servant agreed with a gentleman to serve him twelve months, provided he would give him a farthing for his first month's service, a penny for the second, and 4*d.* for the third, &c. What did his wages amount to?

Ans. 5825*l.* 8*s.* $5\frac{1}{4}$*d.*

3. One *Sessa*, an *Indian*, having first invented the game of chess, shewed it to his prince, who was so delighted with it, that he promised him any reward he should ask; upon which Sessa requested that he might be allowed one grain of wheat for the first square on the chess board, two for the second, and so on, doubling continually, to 64, the whole number of squares. Now, supposing a pint to contain 7680 of those grains, and one quarter to be worth 1*l.* 7*s.* 6*d.*, it is required to compute the value of the whole sum of grains.

Ans. 64481488296*l.*