$$
\begin{array}{l}
a^2-2ab+4b^2)a^4+4a^2b^2+16b^4(a^2+2ab+4b^2 \\
\qquad\qquad a^4-2a^3b+4a^2b^2 \\
\hline
\qquad\qquad\qquad 2a^3b+16b^4 \\
\qquad\qquad\qquad 2a^3b-4a^2b^2+8ab^3 \\
\qquad\qquad\qquad\qquad 16b^4 \\
\qquad\qquad\qquad\qquad 4a^2b^2 \\
\hline
\qquad\qquad\qquad\qquad 4a^2b^2-8ab^3+16b^4 \\
\qquad\qquad\qquad\qquad 4a^2b^2-8ab^3+16b^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad 0.
\end{array}
$$

$$
\begin{array}{l}
a^2-2ab+2b^2)a^4+4b^4(a^2+2ab+2b^2 \\
\qquad\qquad a^4-2a^3b+2a^2b^2 \\
\hline
\qquad\qquad\qquad 2a^3b-2a^2b^2+4b^4 \\
\qquad\qquad\qquad 2a^3b-4a^2b^2+4ab^3 \\
\hline
\qquad\qquad\qquad\qquad 2a^2b^2-4ab^3+4b^4 \\
\qquad\qquad\qquad\qquad 2a^2b^2-4ab^3+4b^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad 0.
\end{array}
$$

$$
\begin{array}{l}
1-2x+x^2)1-5x+10x^2-10x^3+5x^4-x^5(1-3x+3x^2-x^3 \\
\qquad\qquad 1-2x+x^2 \\
\hline
\qquad\qquad -3x+9x^2-10x^3 \\
\qquad\qquad -3x+6x^2-\ 3x^3 \\
\hline
\qquad\qquad\qquad 3x^2-7x^3+5x^4 \\
\qquad\qquad\qquad 3x^2-6x^3+3x^4 \\
\hline
\qquad\qquad\qquad\quad -x^3+2x^4-x^5 \\
\qquad\qquad\qquad\quad -x^3+2x^4-x^5 \\
\hline
\qquad\qquad\qquad\qquad\quad 0.
\end{array}
$$

CHAP. V.

Of the Resolution *of* Fractions *into* Infinite Series *.

289. When the dividend is not divisible by the divisor,

* The Theory of Series is one of the most important in all the mathematics. The series considered in this chapter were dis-

the quotient is expressed, as we have already observed, by a fraction: thus, if we have to divide 1 by $1 - a$, we obtain the fraction $\frac{1}{1-a}$. This, however, does not prevent us from attempting the division according to the rules that have been given, nor from continuing it as far as we please; and we shall not fail thus to find the true quotient, though under different forms.

290. To prove this, let us actually divide the dividend 1 by the divisor $1 - a$, thus:

$$\begin{array}{l} 1-a)\,1 \quad * \quad (1+\frac{a}{1-a} \\ \qquad\;\; 1-a \\ \hline \text{remainder } a \end{array}$$

$$\begin{array}{l} \text{or, } 1-a)\,1 \quad * \quad (1+a+\frac{a^2}{1-a} \\ \qquad\quad\;\; 1-a \\ \hline \qquad\qquad\; a \\ \qquad\qquad\; a-a^2 \\ \hline \text{remainder } a^2 \end{array}$$

To find a greater number of forms, we have only to continue dividing the remainder a^2 by $1 - a$;

$$\begin{array}{l} 1-a)\,a^2 \quad * \quad (a^2+\frac{a^3}{1-a} \\ \qquad\;\; a^2-a^3 \\ \hline \qquad\qquad a^3 \end{array}$$

covered by Mercator, about the middle of the last century; and soon after, Newton discovered those which derived from the extraction of roots, and which are treated of in Chapter XII. of this section. This theory has gradually received improvements from several other distinguished mathematicians. The works of James Bernoulli, and the second part of the "Differential Calculus" of Euler, are the books in which the fullest information is to be obtained on these subjects. There is likewise in the Memoirs of Berlin for 1768, a new method by M. de la Grange for resolving, by means of infinite series, all literal equations of any dimensions whatever. F. T.

$$\text{then, } 1-a)a^3 \quad * \left(a^3 + \frac{a^4}{1-a}\right.$$
$$\frac{a^3 - a^4}{\quad a^4}$$

$$\text{and again, } 1-a)a^4 \quad * \left(a^4 + \frac{a^5}{1-a}\right.$$
$$\frac{a^4 - a^5}{\quad a^5}, \text{ \&c.}$$

291. This shews that the fraction $\frac{1}{1-a}$ may be exhibited under all the following forms:

I. $1 + \frac{a}{1-a}$. II. $1 + a + \frac{a^2}{1-a}$;

III. $1 + a + a^2 + \frac{a^3}{1-a}$. IV. $1+a+a^2+a^3 + \frac{a^4}{1-a}$;

V. $1 + a + a^2 + a^3 + a^4 + \frac{a^5}{1-a}$, &c.

Now, by considering the first of these expressions, which is $1 + \frac{a}{1-a}$, and remembering that 1 is the same as $\frac{1-a}{1-a}$, we have

$$1 + \frac{a}{1-a} = \frac{1-a}{1-a} + \frac{a}{1-a} = \frac{1-a+a}{1-a} = \frac{1}{1-a}.$$

If we follow the same process, with regard to the second expression, $1 + a + \frac{a^2}{1-a}$, that is to say, if we reduce the integral part $1 + a$ to the same denominator, $1 - a$, we shall have $\frac{1-a^2}{1-a}$, to which if we add $+ \frac{a^2}{1-a}$, we shall have $\frac{1-a^2+a^2}{1-a}$, that is to say, $\frac{1}{1-a}$.

In the third expression, $1 + a + a^2 + \frac{a^3}{1-a}$, the integers reduced to the denominator $1 - a$ make $\frac{1-a^3}{1-a}$; and if we add to that the fraction $\frac{a^3}{1-a}$, we have $\frac{1}{1-a}$, as before; therefore all these expressions are equal in value to $\frac{1}{1-a}$, the proposed fraction.

292. This being the case, we may continue the series as far as we please, without being under the necessity of performing any more calculations; and thus we shall have

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7 + \frac{a^8}{1-a};$$

or we might continue this farther, and still go on without end; for which reason it may be said that the proposed fraction has been resolved into an infinite series, which is, $1+a+a^2+a^3+a^4+a^5+a^6+a^7+a^8+a^9+a^{10}+a^{11}+a^{12}$, &c. to infinity: and there are sufficient grounds to maintain, that the value of this infinite series is the same as that of the fraction $\frac{1}{1-a}$.

293. What we have said may at first appear strange; but the consideration of some particular cases will make it easily understood. Let us suppose, in the first place, $a=1$; our series will become $1+1+1+1+1+1+1$, &c; and the fraction $\frac{1}{1-a}$, to which it must be equal, becomes $\frac{1}{0}$. Now, we have before remarked, that $\frac{1}{0}$ is a number infinitely great; which is therefore here confirmed in a satisfactory manner. See Art. 83 and 84.

Again, if we suppose $a = 2$, our series becomes $1+2+4+8+16+32+64$, &c. to infinity, and its value must be the same as $\frac{1}{1-2}$, that is to say $\frac{1}{-1} = -1$; which at first sight will appear absurd. But it must be remarked, that if we wish to stop at any term of the above series, we cannot do so without annexing to it the fraction which remains. Suppose, for example, we were to stop at 64, after having written $1+2+4+8+16+32+64$, we must add the fraction $\frac{128}{1-2}$, or $\frac{128}{-1}$, or -128; we shall therefore have $127-128$, that is in fact -1.

Were we to continue the series without intermission, the fraction would be no longer considered; but, in that case, the series would still go on.

294. These are the considerations which are necessary, when we assume for a numbers greater than unity; but if we suppose a less than 1, the whole becomes more intelligible: for example, let $a = \frac{1}{2}$; and we shall then have $\frac{1}{1-a} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$, which will be equal to the following series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$, &c. to in-

finity. Now, if we take only two terms of this series, we shall have $1 + \frac{1}{2}$, and it wants $\frac{1}{2}$ of being equal to $\frac{1}{1-a} = 2$. If we take three terms, it wants $\frac{1}{4}$; for the sum is $1\frac{3}{4}$. If we take four terms, we have $1\frac{7}{8}$, and the deficiency is only $\frac{1}{8}$. Therefore, the more terms we take, the less the difference becomes; and, consequently, if we continue the series to infinity, there will be no difference at all between its sum and the value of the fraction $\frac{1}{1-a}$, or 2.

295. Let $a = \frac{1}{3}$; and our fraction $\frac{1}{1-a}$ will then be $= \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1\frac{1}{2}$, which, reduced to an infinite series, becomes $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$, &c. which is consequently equal to $\frac{1}{1-a}$.

Here, if we take two terms, we have $1\frac{1}{3}$, and there wants $\frac{1}{6}$. If we take three terms, we have $1\frac{4}{9}$, and there will still be wanting $\frac{1}{18}$. If we take four terms, we shall have $1\frac{13}{27}$, and the difference will be $\frac{1}{54}$; since, therefore, the error always becomes three times less, it must evidently vanish at last.

296. Suppose $a = \frac{2}{3}$; we shall have $\frac{1}{1-a} = \frac{1}{1-\frac{2}{3}} = 3$, $= 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243}$, &c. to infinity; and here, by taking first $1\frac{2}{3}$, the error is $1\frac{1}{3}$; taking three terms, which make $2\frac{1}{9}$, the error is $\frac{8}{9}$; taking four terms, we have $2\frac{11}{27}$, and the error is $\frac{16}{27}$.

297. If $a = \frac{1}{4}$, the fraction is $\frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = 1\frac{1}{3}$; and the series becomes $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$, &c. The first two terms are equal to $1\frac{1}{4}$, which gives $\frac{1}{12}$ for the error; and taking one term more, we have $1\frac{5}{16}$, that is to say, only an error of $\frac{1}{48}$.

298. In the same manner we may resolve the fraction $\frac{1}{1+a}$, into an infinite series by actually dividing the numerator 1 by the denominator $1 + a$, as follows*.

* After a certain number of terms have been obtained, the law by which the following terms are formed will be evident; so that the series may be carried to any length without the trouble of continual division, as is shewn in this example.

$$\begin{array}{l}
1+a)\ 1\ (1-a+a^2-a^3+a^4 \\
\quad 1+a \\
\quad \overline{\quad -a} \\
\quad\quad -a-a^2 \\
\quad\quad \overline{\quad\quad a^2} \\
\quad\quad\quad a^2+a^3 \\
\quad\quad\quad \overline{\quad -a^3} \\
\quad\quad\quad\quad -a^3-a^4 \\
\quad\quad\quad\quad \overline{\quad\quad a^4} \\
\quad\quad\quad\quad\quad a^4+a^5 \\
\quad\quad\quad\quad\quad \overline{\quad -a^5, \text{\&c.}}
\end{array}$$

Whence it follows, that the fraction $\frac{1}{1+a}$ is equal to the series,

$$1 - a + a^2 - a^3 + a^4 - a^5 + a^6 - a^7, \text{\&c.}$$

299. If we make $a = 1$, we have this remarkable comparison:

$\frac{1}{1+a} = \frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1$, &c. to infinity; which appears rather contradictory; for if we stop at -1, the series gives 0; and if we finish at $+1$, it gives 1; but this is precisely what solves the difficulty; for since we must go on to infinity, without stopping either at -1 or at $+1$, it is evident, that the sum can neither be 0 nor 1, but that this result must lie between these two, and therefore be $\frac{1}{2}$*.

300. Let us now make $a = \frac{1}{2}$, and our fraction will be $\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$, which must therefore express the value of the series $1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}$, &c. to infinity; here if we take only the two leading terms of this series, we have $\frac{1}{2}$, which is too small by $\frac{1}{6}$; if we take three terms, we have $\frac{3}{4}$, which is too much by $\frac{1}{12}$; if we take four terms, we have $\frac{5}{8}$, which is too small by $\frac{1}{24}$, &c.

* It may be observed, that no infinite series is in reality equal to the fraction from which it is derived, unless the remainder be considered, which, in the present case, is alternately $+\frac{1}{2}$ and $-\frac{1}{2}$; that is, $+\frac{1}{2}$ when the series is 0, and $-\frac{1}{2}$ when the series is 1, which still gives the same value for the whole expression. Vid. Art. 293.

301. Suppose again $a = \frac{1}{3}$, our fraction will then be $= \dfrac{1}{1+\frac{1}{3}} = \frac{3}{4}$, which must be equal to this series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729}$, &c. continued to infinity. Now, by considering only two terms, we have $\frac{2}{3}$, which is too small by $\frac{1}{12}$; three terms make $\frac{7}{9}$, which is too much by $\frac{1}{36}$; four terms give $\frac{20}{27}$, which is too small by $\frac{1}{108}$, and so on.

302. The fraction $\dfrac{1}{1+a}$ may also be resolved into an infinite series another way; namely, by dividing 1 by $a + 1$, as follows:

$$\begin{array}{l}
a + 1)\ 1 \quad * \ (\dfrac{1}{a} - \dfrac{1}{a^2} + \dfrac{1}{a^3}, \text{ \&c.} \\
\qquad 1 + \dfrac{1}{a} \\
\hline
\qquad\quad - \dfrac{1}{a} \\
\qquad\quad - \dfrac{1}{a} - \dfrac{1}{a^2} \\
\hline
\qquad\qquad\quad \dfrac{1}{a^2} \\
\qquad\qquad\quad \dfrac{1}{a^2} + \dfrac{1}{a^3} \\
\hline
\qquad\qquad\qquad - \dfrac{1}{a^3}, \text{ \&c.*}
\end{array}$$

Consequently, our fraction $\dfrac{1}{a+1}$, is equal to the infinite series $\dfrac{1}{a} - \dfrac{1}{a^2} + \dfrac{1}{a^3} - \dfrac{1}{a^4} + \dfrac{1}{a^5} - \dfrac{1}{a^6}$, &c. Let us make $a = 1$, and we shall have the series $1 - 1 + 1 - 1 + 1 - 1$, &c. $= \frac{1}{2}$, as before: and if we suppose $a = 2$, we shall have the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$, &c. $= \frac{1}{3}$.

* It is unnecessary to carry the actual division any farther, as the series may be continued to any length, from the law observable in the terms already obtained; for the signs are alternately *plus* and *minus*, and any subsequent term may be obtained by multiplying that immediately preceding it by $\dfrac{1}{a}$.

303. In the same manner, by resolving the general fraction $\frac{c}{a+b}$ into an infinite series, we shall have,

$$\begin{array}{lll}
a+b) & c & * \left(\frac{c}{a}-\frac{bc}{a^2}+\frac{b^2c}{a^3}-\frac{b^3c}{a^4}\right. * \\
 & c+\frac{bc}{a} & \\
\hline
 & -\frac{bc}{a} & \\
 & -\frac{bc}{a}-\frac{b^2c}{a^2} & \\
\hline
 & \frac{b^2c}{a^2} & \\
 & \frac{b^2c}{a^2}+\frac{b^3c}{a^3} & \\
\hline
 & -\frac{b^3c}{a^3} &
\end{array}$$

Whence it appears, that we may compare $\frac{c}{a+b}$ with the series $\frac{c}{a}-\frac{bc}{a^2}+\frac{b^2c}{a^3}-\frac{b^3c}{a^4}$, &c. to infinity.

Let $a=2$, $b=4$, $c=3$, and we shall have

$\frac{c}{a+b}=\frac{3}{2+4}=\frac{3}{6}=\frac{1}{2}=\frac{3}{2}-3+6-12$, &c.

If $a=10$, $b=1$, and $c=11$, we shall have

$\frac{c}{a+b}=\frac{11}{10+1}=1=\frac{11}{10}-\frac{11}{100}+\frac{11}{1000}-\frac{11}{10000}$, &c.

Here if we consider only one term of the series, we have $\frac{11}{10}$, which is too much by $\frac{1}{10}$; if we take two terms, we have $\frac{99}{100}$, which is too small by $\frac{1}{100}$; if we take three terms, we have $\frac{1001}{1000}$, which is too much by $\frac{1}{1000}$, &c.

304. When there are more than two terms in the divisor, we may also continue the division to infinity in the same

* Here again the law of continuation is manifest; the signs being alternately + and −, and each succeeding term is formed by multiplying the foregoing one by $\frac{b}{a}$.

manner. Thus, if the fraction $\frac{1}{1-a+a^2}$ were proposed, the infinite series, to which it is equal, will be found as follows:

$$
\begin{array}{l}
1-a+a^2)\ 1 \quad * \quad *\ (1+a,\ \&c. \\
\qquad\qquad 1-a+a^2 \\
\qquad\qquad \overline{\qquad\quad a-a^2} \\
\qquad\qquad\qquad a-a^2+a^3 \\
\qquad\qquad\qquad \overline{\qquad\quad -a^3} \\
\qquad\qquad\qquad\qquad -a^3+a^4-a^5 \\
\qquad\qquad\qquad\qquad \overline{\qquad\quad -a^4+a^5} \\
\qquad\qquad\qquad\qquad\qquad -a^4+a^5-a^6 \\
\qquad\qquad\qquad\qquad\qquad \overline{\qquad\qquad\quad a^6} \\
\qquad\qquad\qquad\qquad\qquad\qquad a^6-a^7+a^8 \\
\qquad\qquad\qquad\qquad\qquad\qquad \overline{\qquad\quad a^7-a^8} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad a^7-a^8+a^9 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\qquad\qquad -a^9}
\end{array}
$$

We have therefore the equation

$\frac{1}{1-a+a^2} = 1 + a - a^3 - a^4 + a^6 + a^7$, &c.; where, if we make $a = 1$, we have $1 = 1 + 1 - 1 - 1 + 1 + 1 - 1 - 1$, &c. which series contains twice the series found above $1 - 1 + 1 - 1 + 1$, &c. Now, as we have found this to be $\frac{1}{2}$, it is not extraordinary that we should find $\frac{2}{2}$, or 1, for the value of that which we have just determined.

By making $a = \frac{1}{2}$, we shall have the equation $\frac{1}{\frac{3}{4}} = \frac{4}{3} = 1 + \frac{1}{2} - \frac{1}{8} - \frac{1}{16} + \frac{1}{64} + \frac{1}{128} - \frac{1}{512}$, &c.

If $a = \frac{1}{3}$, we shall have the equation $\frac{1}{\frac{7}{9}} = \frac{9}{7} = 1 + \frac{1}{3} - \frac{1}{27} - \frac{1}{81} + \frac{1}{729}$, &c. and if we take the four leading terms of this series, we have $\frac{104}{81}$, which is only $\frac{1}{567}$ less than $\frac{9}{7}$.

Suppose again $a = \frac{2}{3}$, we shall have $\frac{1}{\frac{7}{9}} = \frac{9}{7} = 1 + \frac{2}{3} - \frac{8}{27} - \frac{16}{81} + \frac{64}{729}$, &c. This series is therefore equal to the preceding one; and, by subtracting one from the other, we obtain $\frac{1}{3} - \frac{7}{27} - \frac{15}{81} + \frac{63}{729}$, &c. which is necessarily $=0$.

305. The method, which we have here explained, serves to resolve, generally, all fractions into infinite series; which is often found to be of the greatest utility. It is also re-

markable, that an infinite series, though it never ceases, may have a determinate value. It should likewise be observed, that, from this branch of mathematics, inventions of the utmost importance have been derived; on which account the subject deserves to be studied with the greatest attention.

QUESTIONS FOR PRACTICE.

1. Resolve $\frac{ax}{a-x}$ into an infinite series.

$$Ans.\ x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3}, \&c.$$

2. Resolve $\frac{b}{a+x}$ into an infinite series.

$$Ans.\ \frac{b}{a} \times \left(1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} +, \&c.\right)$$

3. Resolve $\frac{a^2}{x+b}$ into an infinite series.

$$Ans.\ \frac{a^2}{x} \times \left(1 - \frac{b}{x} + \frac{b^2}{x^2} - \frac{b^3}{x^3} +, \&c.\right)$$

4. Resolve $\frac{1+x}{1-x}$ into an infinite series.

$$Ans.\ 1 + 2x + 2x^2 + 2x^3 + 2x^4, \&c.$$

5. Resolve $\frac{a^2}{(a+x)^2}$ into an infinite series.

$$Ans.\ 1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3}, \&c.$$

CHAP. VI.

Of the Squares *of* Compound Quantities.

306. When it is required to find the square of a compound quantity, we have only to multiply it by itself, and the product will be the square required.

For example, the square of $a + b$ is found in the following manner: