for if we subtract $6 - 2 + 4$ from $9 - 3 + 2$, we evidently obtain

$$9 - 3 + 2 - 6 + 2 - 4 = 0;$$

for $9 - 3 + 2 = 8$; also, $6 - 2 + 4 = 8$; and $8 - 8 = 0$.

267. Subtraction being therefore subject to no difficulty, we have only to remark, that if there are found in the remainder two or more terms, which are entirely similar with regard to the letters, that remainder may be reduced to an abridged form, by the same rules which we have given in addition.

268. Suppose we have to subtract $a - b$ from $a + b$; that is, to take the difference of two numbers from their sum: we shall then have $(a + b) - (a - b)$; but $a - a = 0$, and $b + b = 2b$; the remainder sought is therefore $2b$; that is to say, the double of the less of the two quantities.

269. The following examples will supply the place of further illustrations:

a^2+ab+b^2	$3a-4b+5c$	$a^3+3a^2b+3ab^2+b^3$	$\sqrt{a}+2\sqrt{b}$
$-a^2+ab+b^2$	$2b+4c-6a$	$a^3-3a^2b+3ab^2-b^3$	$\sqrt{a}-3\sqrt{b}$
$2a^2$.	$9a-6b+c$.	$6a^2b+2b^3$.	$5\sqrt{b}$.

CHAP. III.

Of the Multiplication *of* Compound Quantities.

270. When it is only required to represent multiplication, we put each of the expressions, that are to be multiplied together, within two parentheses, and join them to each other, sometimes without any sign, and sometimes placing the sign $\times$ between them. Thus, for example, to represent the product of the two expressions $a - b + c$ and $d - e + f$, we write

$$(a - b + c) \times (d - e + f)$$

or barely, $\quad (a - b + c) \quad (d - e + f)$

which method of expressing products is much used, because it immediately exhibits the factors of which they are composed.

271. But in order to shew how multiplication is actually performed, we may remark, in the first place, that to multiply, for example, a quantity, such as $a - b + c$, by 2,

each term of it is separately multiplied by that number; so that the product is

$$2a - 2b + 2c.$$

And the same thing takes place with regard to all other numbers; for if d were the number by which it was required to multiply the same expression, we should obtain

$$ad - bd + cd.$$

272. In the last article, we have supposed d to be a positive number; but if the multiplier were a negative number, as $-e$, the rule formerly given must be applied; namely, that unlike signs multiplied together produce $-$, and like signs $+$. Thus we should have

$$-ae + be - ce.$$

273. Now, in order to shew how a quantity, A, is to be multiplied by a compound quantity, $d - e$; let us first consider an example in numbers, supposing that A is to be multiplied by $7 - 3$. Here it is evident, that we are required to take the quadruple of A: for if we first take A seven times, it will then be necessary to subtract 3A from that product.

In general, therefore, if it be required to multiply A by $d - e$, we multiply the quantity A first by d, and then by e, and subtract this last product from the first: whence results $d\mathrm{A} - e\mathrm{A}$.

If we now suppose $\mathrm{A} = a - b$, and that this is the quantity to be multiplied by $d - e$; we shall have

$$\begin{aligned} d\mathrm{A} &= ad - bd \\ e\mathrm{A} &= ae - be \end{aligned}$$

whence $d\mathrm{A} - e\mathrm{A} = ad - bd - ae + be$ is the product required.

274. Since therefore we know accurately the product $(a - b) \times (d - e)$, we shall now exhibit the same example of multiplication under the following form:

$$\begin{array}{l} a - b \\ d - e \\ \hline ad - bd - ae + be. \end{array}$$

Which shews, that we must multiply each term of the upper expression by each term of the lower, and that, with regard to the signs, we must strictly observe the rule before given; a rule which this circumstance would completely confirm, if it admitted of the least doubt.

275. It will be easy, therefore, according to this method, to calculate the following example, which is, to multiply $a + b$ by $a - b$;

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline \end{array}$$

Product a^2-b^2.

276. Now, we may substitute for a and b any numbers whatever; so that the above example will furnish the following theorem; viz. The sum of two numbers, multiplied by their difference, is equal to the difference of the squares of those numbers: which theorem may be expressed thus:

$$(a+b)\times(a-b)=a^2-b^2.$$

And from this another theorem may be derived; namely, The difference of two square numbers is always a product, and divisible both by the sum and by the difference of the roots of those two squares; consequently, the difference of two squares can never be a prime number *.

277. Let us now calculate some other examples:

$$\begin{array}{l} 2a-3 \\ \quad a+2 \\ \hline 2a^2-3a \\ \qquad 4a-6 \\ \hline 2a^2+\ a-6 \\ \hline \end{array} \qquad \begin{array}{l} 4a^2-6a+9 \\ 2a+3 \\ \hline 8a^3-12a^2+18a \\ \qquad 12a^2-18a+27 \\ \hline 8a^3+27 \\ \hline \end{array}$$

$$\begin{array}{l} 3a^2-2ab \\ 2a-4b \\ \hline 6a^3-4a^2b \\ -12a^2b+8ab^2 \\ \hline 6a^3-16a^2b+8ab^2 \\ \hline \end{array} \qquad \begin{array}{l} a^2+ab^3 \\ a^4-a^3b^3 \\ \hline a^6+a^5b^3 \\ \quad -a^5b^3-a^4b^6 \\ \hline a^6-a^4b^6 \\ \hline \end{array}$$

* This theorem is general, except when the difference of the two numbers is only 1, and their sum is a prime; then it is evident that the difference of the two squares will also be a prime: thus, $6^2-5^2=11$, $7^2-6^2=13$, $9^2-8^2=17$, &c.

$$\begin{array}{l}
a^2+2ab+2b^2 \\
a^2-2ab+2b^2 \\
\hline
a^4+2a^3b+2a^2b^2 \\
\quad -2a^3b-4a^2b^2-4ab^3 \\
\qquad\qquad 2a^2b^2+4ab^3+4b^4 \\
\hline
\qquad a^4+b^4 \\
\hline
\end{array}$$

$$\begin{array}{l}
2a^2-3ab-4b^2 \\
3a^2-2ab+b^2 \\
\hline
6a^4-9a^3b-12a^2b^2 \\
\quad -4a^3b+6a^2b^2+8ab^3 \\
\qquad\qquad 2a^2b^2-3ab^3-4b^4 \\
\hline
6a^4-13a^3b-4a^2b^2+5ab^3-4b^4 \\
\hline
\end{array}$$

$$\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc \\
a+b+c \\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc \\
\quad a^2b+b^3+bc^2-ab^2-abc-b^2c \\
\qquad\qquad a^2c+b^2c+c^3-abc-ac^2-bc^2 \\
\hline
a^3-3abc+b^3+c^3 \\
\hline
\end{array}$$

278. When we have more than two quantities to multiply together, it will easily be understood that, after having multiplied two of them together, we must then multiply that product by one of those which remain, and so on: but it is indifferent what order is observed in those multiplications.

Let it be proposed, for example, to find the value, or product, of the four following factors, *viz.*

I.	II.	III.	IV.
$(a+b)$	(a^2+ab+b^2)	$(a-b)$	(a^2-ab+b^2).

1st. The product of the factors I. and II.

$$\begin{array}{l}
a^2+ab+b^2 \\
a+b \\
\hline
a^3+a^2b+ab^2 \\
\quad +a^2b+ab^2+b^3 \\
\hline
a^3+2a^2b+2ab^2+b^3 \\
\hline
\end{array}$$

2d. The product of the factors III. and IV.

$$\begin{array}{l}
a^2-ab+b^2 \\
a-b \\
\hline
a^3-a^2b+ab^2 \\
\quad -a^2b+ab^2-b^3 \\
\hline
a^3-2a^2b+2ab^2-b^3 \\
\hline
\end{array}$$

It remains now to multiply the first product I. II. by this second product III. IV.

$$
\begin{array}{l}
a^3+2a^2b+2ab^2+b^3 \\
a^3-2a^2b+2ab^2-b^3 \\
\hline
a^6+2a^5b+2a^4b^2+a^3b^3 \\
\quad -2a^5b-4a^4b^2-4a^3b^3-2a^2b^4 \\
\qquad\qquad 2a^4b^2+4a^3b^3+4a^2b^4+2ab^5 \\
\qquad\qquad\qquad -a^3b^3-2a^2b^4-2ab^5-b^6 \\
\hline
\qquad\qquad\qquad a^6-b^6 \\
\hline
\end{array}
$$

which is the product required.

279. Now let us resume the same example, but change the order of it, first multiplying the factors I. and III. and then II. and IV. together.

$$
\begin{array}{l}
a+b \\
a-b \\
\hline
a^2+ab \\
\quad -ab-b^2 \\
\hline
a^2-b^2 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
a^2+ab+b^2 \\
a^2-ab+b^2 \\
\hline
a^4+a^3b+a^2b^2 \\
\quad -a^3b-a^2b^2-ab^3 \\
\qquad\qquad a^2b^2+ab^3+b^4 \\
\hline
a^4+a^2b^2+b^4 \\
\hline
\end{array}
$$

Then multiplying the two products I. III. and II. IV.

$$
\begin{array}{l}
a^4+a^2b^2+b^4 \\
a^2-b^2 \\
\hline
a^6+a^4b^2+a^2b^4 \\
\quad -a^4b^2-a^2b^4-b^6 \\
\hline
a^6-b^6 \\
\hline
\end{array}
$$

which is the product required.

280. We may perform this calculation in a manner still more concise, by first multiplying the Ist. factor by the IVth. and then the II^{d}. by the III^{d}.

$$
\begin{array}{l}
a^2-ab+b^2 \\
a+b \\
\hline
a^3-a^2b+ab^2 \\
\qquad a^2b-ab^2+b^3 \\
\hline
\quad a^3+b^3 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
a^2+ab+b^2 \\
a-b \\
\hline
a^3+a^2b+ab^2 \\
\quad -a^2b-ab^2-b^3 \\
\hline
a^3-b^3 \\
\hline
\end{array}
$$

It remains to multiply the product I. IV. by that of II. and III.

$$\begin{array}{l} a^3 + b^3 \\ a^3 - b^3 \\ \hline a^6 + a^3b^3 \\ \quad - a^3b^3 - b^6 \\ \hline a^6 - b^6 \\ \hline \end{array}$$

the same result as before.

281. It will be proper to illustrate this example by a numerical application. For this purpose, let us make $a=3$ and $b=2$, we shall then have $a+b=5$, and $a-b=1$; farther, $a^2=9$, $ab=6$, and $b^2=4$: therefore $a^2+ab+b^2=19$, and $a^2-ab+b^2=7$: so that the product required is that of $5 \times 19 \times 1 \times 7$, which is 665.

Now, $a^6=729$, and $b^6=64$; consequently, the product required is $a^6-b^6=665$, as we have already seen.

CHAP. IV.

Of the Division *of* Compound Quantities.

282. When we wish simply to represent division, we make use of the usual mark of fractions; which is, to write the denominator under the numerator, separating them by a line; or to enclose each quantity between parentheses, placing two points between the divisor and dividend, and a line between them. Thus, if it were required, for example, to divide $a+b$ by $c+d$, we should represent the quotient thus; $\frac{a+b}{c+d}$, according to the former method; and thus,

$$(a+b) \div (c+d)$$

according to the latter, where each expression is read $a+b$ *divided by* $c+d$.

283. When it is required to divide a compound quantity by a simple one, we divide each term separately, as in the following examples:

$$(6a - 8b + 4c) \div 2 = 3a - 4b + 2c$$
$$(a^2 - 2ab) \div a = a - 2b$$
$$(a^3 - 2a^2b + 3ab^2) \div a = a^2 - 2ab + 3b^2$$