CHAP. XIII.

Of the Resolution *of* Negative Powers.

370. We have already shewn, that $\frac{1}{a}$ may be expressed by a^{-1}; we may therefore express $\frac{1}{a+b}$ also by $(a+b)^{-1}$; so that the fraction $\frac{1}{a+b}$ may be considered as a power of $a+b$, namely, that power whose exponent is -1; from which it follows, that the series already found as the value of $(a+b)^n$ extends also to this case.

371. Since, therefore $\frac{1}{a+b}$ is the same as $(a+b)^{-1}$, let us suppose, in the general formula, [Art. 361.] $n = -1$; and we shall first have, for the coefficients, $\frac{n}{1} = -1$; $\frac{n-1}{2} = -1$; $\frac{n-2}{3} = -1$; $\frac{n-3}{4} = -1$, &c. And, for the powers of a, we have $a^n = a^{-1} = \frac{1}{a}$; $a^{n-1} = a^{-2} = \frac{1}{a^2}$; $a^{n-2} = \frac{1}{a^3}$; $a^{n-3} = \frac{1}{a^4}$ &c.: so that $(a+b)^{-1} = \frac{1}{a+b}$ $= \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6}$, &c. which is the same series that we found before by division.

372. Farther, $\frac{1}{(a+b)^2}$ being the same with $(a+b)^{-2}$, let

any degree whatever by approximation; where he demonstrates this general formula,

$$\sqrt[m]{(a^m \pm b)} = \frac{m-2}{m-1}a + \sqrt{\left(\frac{a^2}{(m-1)^2} \pm \frac{2b}{(m^2-m)a^{m-1}}\right)}.$$

Those who have not an opportunity of consulting the Philosophical Transactions, will find the formation and the use of this formula explained in the new edition of Leçons Elementaires de Mathematiques by M. D'Abbé de la Caille, published by M. L'Abbé Marie. F. T. See also Dr. Hutton's Math. Dictionary.

us reduce this quantity also to an infinite series. For this purpose, we must suppose $n = -2$, and we shall first have, for the coefficients, $\frac{n}{1} = -\frac{2}{1}$; $\frac{n-1}{2} = -\frac{3}{2}$; $\frac{n-2}{3} = -\frac{4}{3}$; $\frac{n-3}{4} = -\frac{5}{4}$, &c.; and, for the powers of a, we obtain $a^n = \frac{1}{a^2}$; $a^{n-1} = \frac{1}{a^3}$; $a^{n-2} = \frac{1}{a^4}$; $a^{n-3} = \frac{1}{a^5}$, &c. We have therefore $(a+b)^{-2} = \frac{1}{(a+b)^2} = \frac{1}{a^2} - \frac{2.b}{1.a^3} + \frac{2.3.b^2}{1.2.a^4} - \frac{2.3.4.b^3}{1.2.3.a^5} + \frac{2.3.4.5.b^4}{1.2.3.4.a^6}$, &c. Now, $\frac{2}{1} = 2$; $\frac{2.3}{1.2} = 3$; $\frac{2.3.4}{1.2.3} = 4$; $\frac{2.3.4.5}{1.2.3.4} = 5$, &c. and consequently, $\frac{1}{(a+b)^2} = \frac{1}{a^2} - 2\frac{b}{a^3} + 3\frac{b^2}{a^4} - 4\frac{b^3}{a^5} + 5\frac{b^4}{a^6} - 6\frac{b^5}{a^7} + 7\frac{b^6}{a^8}$, &c.

373. Let us proceed, and suppose $n = -3$, and we shall have a series expressing the value of $\frac{1}{(a+b)^3}$, or of $(a+b)^{-3}$. Here the coefficients will be $\frac{n}{1} = -\frac{3}{1}$; $\frac{n-1}{2} = -\frac{4}{2}$; $\frac{n-2}{3} = -\frac{5}{3}$, &c. and the powers of a become, $a^n = \frac{1}{a^3}$; $a^{-1} = \frac{1}{a^4}$; $a^{n-2} = \frac{1}{a^5}$, &c. which gives $\frac{1}{(a+b)^3} = \frac{1}{a^3} - \frac{3.b}{1.a^4} + \frac{3.4.b^2}{1.2.a^5} - \frac{3.4.5.b^3}{1.2.3.a^6} + \frac{3.4.5.6.b^4}{1.2.3.4.a^7} = \frac{1}{a^3} - 3\frac{b}{a^4} + 6\frac{b^2}{a^5} - 10\frac{b^3}{a^6} + 15\frac{b^4}{a^7} - 21\frac{b^5}{a^8} + 28\frac{b^6}{a^9}$, &c.

If now we make $n = -4$; we shall have for the coefficients $\frac{n}{1} = -\frac{4}{1}$; $\frac{n-1}{2} = -\frac{5}{2}$; $\frac{n-2}{3} = -\frac{6}{3}$; $\frac{n-3}{4} = -\frac{7}{4}$, &c. And for the powers, $a^n = \frac{1}{a^4}$; $a^{n-1} = \frac{1}{a^5}$; $a^{n-2} = \frac{1}{a^6}$; $a^{n-3} = \frac{1}{a^7}$; $a^{n-4} = \frac{1}{a^8}$, whence we obtain,

$$\frac{1}{(a+b)^4} = \frac{1}{a^4} - \frac{4b}{1a^5} + \frac{4.5.b^2}{1.2.a^6} - \frac{4.5.6.b^3}{1.2.3.a^7}, \text{ \&c.} = \frac{1}{a^4} - 4\frac{b}{a^5} + 10\frac{b^2}{a^6} - 20\frac{b^3}{a^7} + 35\frac{b^4}{a^8} - 56\frac{b^5}{a^9} +, \text{ \&c.}$$

374. The different cases that have been considered

enable us to conclude with certainty, that we shall have, generally, for any negative power of $a+b$;

$$\frac{1}{(a+b)^m}=\frac{1}{a^m}-\frac{m.b}{a^{m+1}}+\frac{m.(m-1).b^2}{2.a^{m+2}}-\frac{m.(m-1).(m-2).b^3}{2.3.a^{m+3}},$$

&c. And, by means of this formula, we may transform all such fractions into infinite series, substituting fractions also, or fractional exponents, for m, in order to express irrational quantities.

375. The following considerations will illustrate this subject still farther: for we have seen that,

$$\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}-\frac{b^5}{a^6}+, \text{\&c.}$$

If, therefore, we multiply this series by $a+b$, the product ought to be $=1$; and this is found to be true, as will be seen by performing the multiplication:

$$\begin{array}{l}
\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}-\frac{b^5}{a^6}+, \text{\&c.}\\
a+b\\
\hline
1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\frac{b^4}{a^4}-\frac{b^5}{a^5}+, \text{\&c.}\\
\quad+\frac{b}{a}-\frac{b^2}{a^2}+\frac{b^3}{a^3}-\frac{b^4}{a^4}+\frac{b^5}{a^5}-, \text{\&c.}\\
\hline
\end{array}$$

where all the terms but the first cancel each other.

376. We have also found, that

$$\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-\frac{6b^5}{a^7}, \text{\&c.}$$

And if we multiply this series by $(a+b)^2$, the product ought also to be equal to 1. Now, $(a+b)^2=a^2+2ab+b^2$, and

$$\begin{array}{l}
\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-\frac{6b^5}{a^7}+a, \text{\&c.}\\
a^2+2ab+b^2\\
\hline
1-\frac{2b}{a}+\frac{3b^2}{a^2}-\frac{4b^3}{a^3}+\frac{5b^4}{a^4}-\frac{6b^5}{a^5}+, \text{\&c.}\\
\quad+\frac{2b}{a}-\frac{4b^2}{a^2}+\frac{6b^3}{a^3}-\frac{8b^4}{a^4}+\frac{10b^5}{a^5}-, \text{\&c.}\\
\qquad+\frac{b^2}{a^2}-\frac{2b^3}{a^3}+\frac{3b^4}{a^4}-\frac{4b^5}{a^5}+, \text{\&c.}
\end{array}$$

which gives 1 for the product, as the nature of the thing required.

377. If we multiply the series which we found for the value of $\frac{1}{(a+b)^2}$, by $a+b$ only, the product ought to answer to the fraction $\frac{1}{a+b}$, or be equal to the series already found, namely, $\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}$, &c. and this the actual multiplication will confirm.

$$\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}, \text{ \&c.}$$
$$a+b$$

$$\frac{1}{a}-\frac{2b}{a^2}+\frac{3b^2}{a^3}-\frac{4b^3}{a^4}+\frac{5b^4}{a^5}, \text{ \&c.}$$
$$+\frac{b}{a^2}-\frac{2b^2}{a^3}+\frac{3b^3}{a^4}-\frac{4b^4}{a^5}, \text{ \&c.}$$

$$\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}, \text{ \&c. as required.}$$

SECTION III.

Of Ratios *and* Proportions.

CHAP. I.

Of Arithmetical Ratio, *or of the* Difference *between two* Numbers.

378. Two quantities are either equal to one another, or they are not. In the latter case, where one is greater than the other, we may consider their inequality under two different points of view: we may ask, *how much* one of the quantities is greater than the other? Or we may ask, *how many times* the one is greater than the other? The