CHAP. VI.

Of the Properties *of* Integers, *with respect to their* Divisors.

58. As we have seen that some numbers are divisible by certain divisors, while others are not so; it will be proper, in order that we may obtain a more particular knowledge of numbers, that this difference should be carefully observed, both by distinguishing the numbers that are divisible by divisors from those which are not, and by considering the remainder that is left in the division of the latter. For this purpose let us examine the divisors;

2, 3, 4, 5, 6, 7, 8, 9, 10, &c.

59. First let the divisor be 2; the numbers divisible by it are, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, &c. which, it appears, increase always by two. These numbers, as far as they can be continued, are called *even numbers*. But there are other numbers, viz.

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, &c.

which are uniformly less or greater than the former by unity, and which cannot be divided by 2, without the remainder 1; these are called *odd numbers*.

The even numbers are all comprehended in the general expression $2a$; for they are all obtained by successively substituting for a the integers 1, 2, 3, 4, 5, 6, 7, &c. and hence it follows that the odd numbers are all comprehended in the expression $2a + 1$, because $2a + 1$ is greater by unity than the even number $2a$.

60. In the second place, let the number 3 be the divisor; the numbers divisible by it are,

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, and so on;

which numbers may be represented by the expression $3a$; for $3a$, divided by 3, gives the quotient a without a remainder. All other numbers which we would divide by 3, will give 1 or 2 for a remainder, and are consequently of two kinds. Those which after the division leave the remainder 1, are,

1, 4, 7, 10, 13, 16, 19, &c.

and are contained in the expression $3a + 1$; but the other kind, where the numbers give the remainder 2, are,

2, 5, 8, 11; 14, 17, 20, &c.

which may be generally represented by $3a + 2$; so that all numbers may be expressed either by $3a$, or by $3a + 1$, or by $3a + 2$.

61. Let us now suppose that 4 is the divisor under consideration; then the numbers which it divides are,

4, 8, 12, 16, 20, 24, &c.

which increase uniformly by 4, and are comprehended in the expression $4a$. All other numbers, that is, those which are not divisible by 4, may either leave the remainder 1, or be greater than the former by 1; as,

1, 5, 9, 13, 17, 21, 25, &c.

and consequently may be comprehended in the expression $4a + 1$: or they may give the remainder 2; as,

2, 6, 10, 14, 18, 22, 26, &c.

and be expressed by $4a + 2$; or, lastly, they may give the remainder 3; as,

3, 7, 11, 15, 19, 23, 27, &c.

and may then be represented by the expression $4a + 3$.

All possible integer numbers are contained therefore in one or other of these four expressions;

$$4a,\ 4a + 1,\ 4a + 2,\ 4a + 3.$$

62. It is also nearly the same when the divisor is 5; for all numbers which can be divided by it are comprehended in the expression $5a$, and those which cannot be divided by 5, are reducible to one of the following expressions:

$$5a + 1,\ 5a + 2,\ 5a + 3,\ 5a + 4;$$

and in the same manner we may continue, and consider any greater divisor.

63. It is here proper to recollect what has been already said on the resolution of numbers into their simple factors; for every number, among the factors of which is found

2, or 3, or 4, or 5, or 7,

or any other number, will be divisible by those numbers. For example; 60 being equal to $2 \times 2 \times 3 \times 5$, it is evident that 60 is divisible by 2, and by 3, and by 5 *.

* There are some numbers which it is easy to perceive whether they are divisors of a given number or not.

1. A given number is divisible by 2, if the last digit is even; it is divisible by 4, if the two last digits are divisible by 4; it is divisible by 8, if the three last digits are divisible by 8; and, in general, it is divisible by 2^n, if the n last digits are divisible by 2^n.

2. A number is divisible by 3, if the sum of the digits is divisible by 3; it may be divided by 6, if, beside this, the last digit is even; it is divisible by 9, if the sum of the digits may be divided by 9.

3. Every number that has the last digit 0 or 5, is divisible by 5.

64. Farther, as the general expression *abcd* is not only divisible by *a*, and *b*, and *c*, and *d*, but also by

ab, *ac*, *ad*, *bc*, *bd*, *cd*, and by
abc, *abd*, *acd*, *bcd*, and lastly by
abcd, that is to say, its own value;

it follows that 60, or $2 \times 2 \times 3 \times 5$, may be divided not only by these simple numbers, but also by those which are composed of any two of them; that is to say, by 4, 6, 10, 15: and also by those which are composed of any three of its simple factors; that is to say, by 12, 20, 30, and lastly also, by 60 itself.

65. When, therefore, we have represented any number, assumed at pleasure, by its simple factors, it will be very easy to exhibit all the numbers by which it is divisible. For we have only, first, to take the simple factors one by one, and then to multiply them together two by two,

4. A number is divisible by 11, when the sum of the first third, fifth, &c. digits is equal to the sum of the second, fourth, sixth, &c. digits.

It would be easy to explain the reason of these rules, and to extend them to the products of the divisors which we have just now considered. Rules might be devised likewise for some other numbers, but the application of them would in general be longer than an actual trial of the division.

For example, I say that the number 53704689213 is divisible by 7, because I find that the sum of the digits of the number 64004245433 is divisible by 7; and this second number is formed, according to a very simple rule, from the remainders found after dividing the component parts of the former number by 7.

Thus, $53704689213 = 50000000000 + 3000000000 + 700000000 + 0 + 4000000 + 600000 + 80000 + 9000 + 200 + 10 + 3$; which being, each of them, divided by 7, will leave the remainders 6, 4, 0, 0, 4, 2, 4, 5, 4, 3, 3, the number here given.

If *a*, *b*, *c*, *d*, *e*, &c. be the digits composing any number, the number itself may be expressed universally thus; $a + 10b + 10^2c + 10^3d + 10^4e$, &c. to 10^nz; where it is easy to perceive that, if each of the terms a, $10b$, 10^2c, &c. be divisible by n, the number itself $a + 10b + 10^2c$, &c. will also be divisible by n. And, if $\frac{a}{n}$, $\frac{10b}{n}$, $\frac{10^2c}{n}$, &c. leave the remainders p, q, r, &c. it is obvious, that $a + 10b + 10^2c$, &c. will be divisible by n, when $p + q + r$, is divisible by n; which renders the principle of the rule sufficiently clear.

The reader is indebted to that excellent mathematician, the late Professor Bonnycastle, for this satisfactory illustration of M. Bernoulli's note.

three by three, four by four, &c. till we arrive at the number proposed.

66. It must here be particularly observed, that every number is divisible by 1; and also, that every number is divisible by itself; so that every number has at least two factors, or divisors, the number itself, and unity: but every number which has no other divisor than these two, belongs to the class of numbers, which we have before called *simple*, or *prime numbers*.

Except these simple numbers, all other numbers have, beside unity and themselves, other divisors, as may be seen from the following Table, in which are placed under each number all its divisors*.

TABLE.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	3	2	5	2	7	2	3	2	11	2	13	2	3	2	17	2	19	2
			4		3		4	9	5		3		7	5	4		3		4
					6		8		10		4		14	15	8		6		5
											6				16		9		10
											12						18		20
1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6
P.	P.	P.		P.		P.				P.		P.				P.		P.	

67. Lastly, it ought to be observed that 0, or *nothing*, may be considered as a number which has the property of being divisible by all possible numbers; because by whatever number a we divide 0, the quotient is always 0; for it must be remarked, that the multiplication of any number by *nothing* produces nothing, and therefore 0 times a, or $0a$, is 0.

* A similar Table for all the divisors of the natural numbers, from 1 to 10000, was published at Leyden, in 1767, by M. Henry Anjema. We have likewise another table of divisors, which goes as far as 100000, but it gives only the least divisor of each number. It is to be found in Harris's Lexicon Technicum, the Encyclopédie, and in M. Lambert's Recueil, which we have quoted in the note to p. 11. In this last work, it is continued as far as 102000. F. T.