is $\frac{27}{8}$, or $3\frac{3}{8}$; also the cube of $1\frac{1}{4}$, or of the single fraction $\frac{5}{4}$, is $\frac{125}{64}$, or $1\frac{61}{64}$; and the cube of $3\frac{1}{4}$, or of $\frac{13}{4}$, is $\frac{2197}{64}$, or $34\frac{21}{64}$.

156. Since aaa is the cube of a, that of ab will be $aaabbb$; whence we see, that if a number has two or more factors, we may find its cube by multiplying together the cubes of those factors. For example, as 12 is equal to 3×4, we multiply the cube of 3, which is 27, by the cube of 4, which is 64, and we obtain 1728, the cube of 12; and farther, the cube of $2a$ is $8aaa$, and consequently 8 times greater than the cube of a: likewise, the cube of $3a$ is $27aaa$; that is to say, 27 times greater than the cube of a.

157. Let us attend here also to the signs $+$ and $-$. It is evident that the cube of a positive number $+a$ must also be positive, that is $+aaa$; but if it be required to cube a negative number $-a$, it is found by first taking the square, which is $+aa$, and then multiplying, according to the rule, this square by $-a$, which gives for the cube required $-aaa$. In this respect, therefore, it is not the same with cubic numbers as with squares, since the latter are always positive: whereas the cube of -1 is -1, that of -2 is -8, that of -3 is -27, and so on.

CHAP. XV.

Of Cube Roots, *and of* Irrational Numbers *resulting from them.*

158. As we can, in the manner already explained, find the cube of a given number, so, when a number is proposed, we may also reciprocally find a number, which, multiplied twice by itself, will produce that number. The number here sought is called, with relation to the other, *the cube root;* so that the cube root of a given number is the number whose cube is equal to that given number.

159. It is easy therefore to determine the cube root, when the number proposed is a real cube, such as in the examples in the last chapter; for we easily perceive that the cube root of 1 is 1; that of 8 is 2; that of 27 is 3; that of 64 is 4, and so on. And, in the same manner, the cube root of -27 is -3; and that of -125 is -5.

Farther, if the proposed number be a fraction, as $\frac{8}{27}$, the cube root of it must be $\frac{2}{3}$; and that of $\frac{64}{343}$ is $\frac{4}{7}$. Lastly, the cube root of a mixed number, such as $2\frac{10}{27}$ must be $\frac{4}{3}$, or $1\frac{1}{3}$; because $2\frac{10}{27}$ is equal to $\frac{64}{27}$.

160. But if the proposed number be not a cube, its cube root cannot be expressed either in integers, or in fractional numbers. For example, 43 is not a cubic number; therefore it is impossible to assign any number, either integer or fractional, whose cube shall be exactly 43. We may however affirm, that the cube root of that number is greater than 3, since the cube of 3 is only 27; and less than 4, because the cube of 4 is 64: we know, therefore, that the cube root required is necessarily contained between the numbers 3 and 4.

161. Since the cube root of 43 is greater than 3, if we add a fraction to 3, it is certain that we may approximate still nearer and nearer to the true value of this root: but we can never assign the number which expresses the value exactly; because the cube of a mixed number can never be perfectly equal to an integer, such as 43. If we were to suppose, for example, $3\frac{1}{2}$, or $\frac{7}{2}$ to be the cube root required, the error would be $\frac{1}{8}$; for the cube of $\frac{7}{2}$ is only $\frac{343}{8}$, or $42\frac{7}{8}$.

162. This therefore shews, that the cube root of 43 cannot be expressed in any way, either by integers or by fractions. However, we have a distinct idea of the magnitude of this root; and therefore we use, in order to represent it, the sign $\sqrt[3]{\ }$, which we place before the proposed number, and which is read *cube root*, to distinguish it from the square root, which is often called simply *the root;* thus $\sqrt[3]{43}$ means the cube root of 43; that is to say, the number whose cube is 43, or which, multiplied by itself, and then by itself again, produces 43.

163. Now, it is evident that such expressions cannot belong to rational quantities, but that they rather form a particular species of irrational quantities. They have nothing in common with square roots, and it is not possible to express such a cube root by a square root; as, for example, by $\sqrt{12}$; for the square of $\sqrt{12}$ being 12, its cube will be $12\sqrt{12}$, consequently still irrational, and therefore it cannot be equal to 43.

164. If the proposed number be a real cube, our expressions become rational. Thus, $\sqrt[3]{1}$ is equal to 1; $\sqrt[3]{8}$ is equal to 2; $\sqrt[3]{27}$ is equal to 3; and, generally, $\sqrt[3]{aaa}$ is equal to a.

165. If it were proposed to multiply one cube root, $\sqrt[3]{a}$, by another, $\sqrt[3]{b}$, the product must be $\sqrt[3]{ab}$; for we know that

the cube root of a product ab is found by multiplying together the cube roots of the factors. Hence, also, if we divide $\sqrt[3]{a}$ by $\sqrt[3]{b}$, the quotient will be $\sqrt[3]{\frac{a}{b}}$.

166. We farther perceive, that $2\sqrt[3]{a}$ is equal to $\sqrt[3]{8a}$, because 2 is equivalent to $\sqrt[3]{8}$; that $3\sqrt[3]{a}$ is equal to $\sqrt[3]{27a}$, $b\sqrt[3]{a}$ is equal to $\sqrt[3]{abbb}$; and, reciprocally, if the number under the radical sign has a factor which is a cube, we may make it disappear by placing its cube root before the sign; for example, instead of $\sqrt[3]{64a}$ we may write $4\sqrt[3]{a}$; and $5\sqrt[3]{a}$ instead of $\sqrt[3]{125a}$: hence $\sqrt[3]{16}$ is equal to $2\sqrt[3]{2}$, because 16 is equal to 8×2.

167. When a number proposed is negative, its cube root is not subject to the same difficulties that occurred in treating of square roots; for, since the cubes of negative numbers are negative, it follows that the cube roots of negative numbers are also negative; thus $\sqrt[3]{-8}$ is equal to -2, and $\sqrt[3]{-27}$ to -3. It follows also, that $\sqrt[3]{-12}$ is the same as $-\sqrt[3]{12}$, and that $\sqrt[3]{-a}$ may be expressed by $-\sqrt[3]{a}$. Whence we see that the sign $-$, when it is found after the sign of the cube root, might also have been placed before it. We are not therefore led here to impossible, or imaginary numbers, which happened in considering the square roots of negative numbers.

CHAP. XVI.

Of Powers *in general.*

168. The product which we obtain by multiplying a number once, or several times by itself, is called *a power*. Thus, a square which arises from the multiplication of a number by itself, and a cube which we obtain by multiplying a number twice by itself, are powers. We say also in the former case, that the number is raised to the second degree, or to the second power; and in the latter, that the number is raised to the third degree, or to the third power.

169. We distinguish those powers from one another by the number of times that the given number has been multiplied by itself. For example, a square is called the second