CHAP. XI.

Of Square Numbers.

115. The product of a number, when multiplied by itself, is called *a square*; and, for this reason, the number, considered in relation to such a product, is called *a square root*. For example, when we multiply 12 by 12, the product 144 is a square, of which the root is 12.

The origin of this term is borrowed from geometry, which teaches us that the contents of a square are found by multiplying its side by itself.

116. Square numbers are found therefore by multiplication; that is to say, by multiplying the root by itself: thus, 1 is the square of 1, since 1 multiplied by 1 makes 1; likewise, 4 is the square of 2; and 9 the square of 3; 2 also is the root of 4, and 3 is the root of 9.

We shall begin by considering the squares of natural numbers; and for this purpose shall give the following small Table, on the first line of which several numbers, or roots, are ranged, and on the second their squares *.

Numbers.	1	2	3	4	5	6	7	8	9	10	11	12	13
Squares.	1	4	9	16	25	36	49	64	81	100	121	144	169

117. Here it will be readily perceived that the series of square numbers thus arranged has a singular property; namely, that if each of them be subtracted from that which immediately follows, the remainders always increase by 2, and form this series;

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c.

which is that of the odd numbers.

118. The squares of fractions are found in the same manner, by multiplying any given fraction by itself. For example, the square of $\frac{1}{2}$ is $\frac{1}{4}$,

* We have very complete tables for the squares of natural numbers, published under the title "Tetragonometria Tabularia, &c. Auct. J. Jobo Ludolfo, Amstelodami, 1690, in 4to." These Tables are continued from 1 to 100000, not only for finding those squares, but also the products of any two numbers less than 100000; not to mention several other uses, which are explained in the introduction to the work. F. T.

The square of $\left\{\begin{array}{c}\frac{1}{3}\\ \frac{2}{3}\\ \frac{1}{4}\\ \frac{3}{4}\end{array}\right\}$ is $\begin{array}{l}\frac{1}{9};\\ \frac{4}{9};\\ \frac{1}{16};\\ \frac{9}{16}, \text{\&c.}\end{array}$

We have only therefore to divide the square of the numerator by the square of the denominator, and the fraction which expresses that division will be the square of the given fraction; thus, $\frac{25}{64}$ is the square of $\frac{5}{8}$; and reciprocally, $\frac{5}{8}$ is the root of $\frac{25}{64}$.

119. When the square of a mixed number, or a number composed of an integer and a fraction, is required, we have only to reduce it to a single fraction, and then take the square of that fraction. Let it be required, for example, to find the square of $2\frac{1}{2}$; we first express this number by $\frac{5}{2}$, and taking the square of that fraction, we have $\frac{25}{4}$, or $6\frac{1}{4}$, for the value of the square of $2\frac{1}{2}$. Also to obtain the square of $3\frac{1}{4}$, we say $3\frac{1}{4}$ is equal to $\frac{13}{4}$; therefore its square is equal to $\frac{169}{16}$, or to $10\frac{9}{16}$. The squares of the numbers between 3 and 4, supposing them to increase by one fourth, are as follow:

Numbers.	3	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4
Squares.	9	$10\frac{9}{16}$	$12\frac{1}{4}$	$14\frac{1}{16}$	16

From this small Table we may infer, that if a root contain a fraction, its square also contains one. Let the root, for example, be $1\frac{5}{12}$; its square is $\frac{289}{144}$, or $2\frac{1}{144}$; that is to say, a little greater than the integer 2.

120. Let us now proceed to general expressions. First, when the root is a, the square must be aa; if the root be $2a$, the square is $4aa$; which shews that by doubling the root, the square becomes 4 times greater; also, if the root be $3a$, the square is $9aa$; and if the root be $4a$, the square is $16aa$. Farther, if the root be ab, the square is $aabb$; and if the root be abc, the square is $aabbcc$; or $a^2b^2c^2$.

121. Thus, when the root is composed of two, or more factors, we multiply their squares together; and reciprocally, if a square be composed of two, or more factors, of which each is a square, we have only to multiply together the roots of those squares, to obtain the complete root of the square proposed. Thus, 2304 is equal to $4 \times 16 \times 36$, the square root of which is $2 \times 4 \times 6$, or 48; and 48 is found to be the true square root of 2304, because 48×48 gives 2304.

122. Let us now consider what must be observed on this subject with regard to the signs $+$ and $-$. First, it is

evident that if the root have the sign +, that is to say, if it be a positive number, its square must necessarily be a positive number also, because + multiplied by + makes +: hence the square of $+a$ will be $+aa$: but if the root be a negative number, as $-a$, the square is still positive, for it is $+aa$. We may therefore conclude, that $+aa$ is the square both of $+a$ and of $-a$, and that consequently every square has two roots, one positive, and the other negative. The square root of 25, for example, is both +5 and −5, because −5 multiplied by −5 gives 25, as well as +5 by +5.

CHAP. XII.

Of Square Roots, *and of* Irrational Numbers *resulting from them.*

123. What we have said in the preceding chapter amounts to this; that the square root of a given number is that number whose square is equal to the given number; and that we may put before those roots either the positive, or the negative sign.

124. So that when a square number is given, provided we retain in our memory a sufficient number of square numbers, it is easy to find its root. If 196, for example, be the given number, we know that its square root is 14.

Fractions, likewise, are easily managed in the same way. It is evident, for example, that $\frac{5}{7}$ is the square root of $\frac{25}{49}$; to be convinced of which, we have only to take the square root of the numerator and that of the denominator.

If the number proposed be a mixed number, as $12\frac{1}{4}$, we reduce it to a single fraction, which, in this case, will be $\frac{49}{4}$; and from this we immediately perceive that $\frac{7}{2}$, or $3\frac{1}{2}$, must be the square root of $12\frac{1}{4}$.

125. But when the given number is not a square, as 12, for example, it is not possible to extract its square root; or to find a number, which, multiplied by itself, will give the product 12. We know, however, that the square root of 12 must be greater than 3, because 3 × 3 produces only 9; and less than 4, because 4 × 4 produces 16, which is more than 12; we know also, that this root is less than $3\frac{1}{2}$, for we have seen that the square of $3\frac{1}{2}$, or $\frac{7}{2}$, is $12\frac{1}{4}$; and we may approach still nearer to this root, by comparing it with $3\frac{7}{15}$; for the square of $3\frac{7}{15}$, or of $\frac{52}{15}$, is $\frac{2704}{225}$, or $12\frac{4}{225}$; so that this