Zygmund's Fourier restriction theorem and Bernstein's inequality

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1 Zygmund's restriction theorem

Write $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Write λ_d for the Haar measure on $\mathbb{T}^d$ for which $\lambda_d(\mathbb{T}^d) = 1$. For $\xi \in \mathbb{Z}^d$, we define $e_\xi : \mathbb{T}^d \to S^1$ by

$$e_\xi(x) = e^{2\pi i \xi \cdot x}, \qquad x \in \mathbb{T}^d.$$

For $f \in L^1(\mathbb{T}^d)$, we define its **Fourier transform** $\hat{f} : \mathbb{Z}^d \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{\mathbb{T}^d} f \overline{e_\xi} d\lambda_d = \int_{\mathbb{T}^d} f(x) e^{-2\pi i \xi \cdot x} dx, \qquad \xi \in \mathbb{Z}^d.$$

For $x \in \mathbb{R}^d$, we write $|x| = |x|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$, $|x|_1 = |x_1| + \cdots + |x_d|$, and $|x|_\infty = \max\{|x_j| : 1 \leq j \leq d\}$.

For $1 \leq p < \infty$, we write

$$\|f\|_p = \left(\int_{\mathbb{T}^d} |f(x)|^p dx \right)^{1/p}.$$

For $1 \leq p \leq q \leq \infty$, $\|f\|_p \leq \|f\|_q$.

Parseval's identity tells us that for $f \in L^2(\mathbb{T}^d)$,

$$\left\| \hat{f} \right\|_{\ell^2} = \left(\sum_{\xi \in \mathbb{Z}^d} |\hat{f}(\xi)|^2 \right)^{1/2} = \|f\|_2,$$

and the Hausdorff-Young inequality tells us that for $1 \leq p \leq 2$ and $f \in L^p(\mathbb{T}^d)$,

$$\left\| \hat{f} \right\|_{\ell^q} = \left(\sum_{\xi \in \mathbb{Z}^d} |\hat{f}(\xi)|^q \right)^{1/q} \leq \|f\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$; $\left\| \hat{f} \right\|_{\ell^\infty} = \sup_{\xi \in \mathbb{Z}^d} |\hat{f}(\xi)|$.

Zygmund's theorem is the following.[1]

[1] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 236, Theorem 4.3.11.

Theorem 1 (Zygmund's theorem)**.** *For $f \in L^{4/3}(\mathbb{T}^2)$ and $r > 0$,*

$$\left(\sum_{|\xi|=r} |\hat{f}(\xi)|^2 \right)^{1/2} \leq 5^{1/4} \|f\|_{4/3} . \tag{1}$$

Proof. Suppose that

$$S = \left(\sum_{|\xi|=r} |\hat{f}(\xi)|^2 \right)^{1/2} > 0.$$

For $\xi \in \mathbb{Z}^2$, we define

$$c_\xi = \frac{\overline{\hat{f}(\xi)}}{S} \chi_{|\zeta|=r}.$$

Then

$$\sum_{|\xi|=r} |c_\xi|^2 = \sum_{|\xi|=r} \frac{|\hat{f}(\xi)|^2}{|S|^2} = 1. \tag{2}$$

We have

$$\begin{aligned} S^2 &= \sum_{|\xi|=r} |\hat{f}(\xi)|^2 \\ &= \sum_{|\xi|=r} \hat{f}(\xi)\overline{\hat{f}(\xi)} \\ &= \left(\sum_{|\xi|=r} \hat{f}(\xi) c_\xi \right) S, \end{aligned}$$

hence, defining $c : \mathbb{T}^2 \to \mathbb{C}$ by

$$c(x) = \sum_{\xi \in \mathbb{Z}^d} c_\xi e^{2\pi i \xi \cdot x} = \sum_{|\xi|=r} c_\xi e^{2\pi i \xi \cdot x}, \qquad x \in \mathbb{T}^2,$$

we have, applying Parseval's identity,

$$S = \sum_{|\xi|=r} \hat{f}(\xi) c_\xi = \int_{\mathbb{T}^2} f(x)\overline{c(x)} dx.$$

For $p = \frac{4}{3}$, let $\frac{1}{p} + \frac{1}{q} = 1$, i.e. $q = 4$. Hölder's inequality tells us

$$\int_{\mathbb{T}^2} |f(x)\overline{c(x)}| dx \leq \|f\|_{4/3} \|c\|_4 .$$

For $\rho \in \mathbb{Z}^2$, we define

$$\gamma_\rho = \sum_{\mu - \nu = \rho} c_\mu \overline{c_\nu}.$$

Then define $\Gamma(x) = |c(x)|^2$, which satisfies

$$\Gamma(x) = c(x)\overline{c(x)} = \sum_{\xi \in \mathbb{Z}^2} \sum_{\zeta \in \mathbb{Z}^2} c_\xi \overline{c_\zeta} e^{2\pi i(\xi - \zeta)\cdot x} = \sum_{\rho \in \mathbb{Z}^2} \gamma_\rho e^{2\pi i \rho \cdot x}.$$

Parseval's identity tells us

$$\|c\|_4^4 = \|\Gamma\|_2^2 = \sum_{\rho \in \mathbb{Z}^2} |\gamma_\rho|^2.$$

First,

$$\gamma_0 = \sum_{\mu \in \mathbb{Z}^2} c_\mu \overline{c_\mu} = \sum_{\mu \in \mathbb{Z}^2} |c_\mu|^2 = 1.$$

Second, suppose that $\rho \in \mathbb{Z}^2, |\rho| = 2r$. If $\rho/2 \in \mathbb{Z}^2$, then $\gamma_\rho = c_{\rho/2}\overline{c_{-\rho/2}}$, and if $\rho/2 \notin \mathbb{Z}^2$ then $\gamma_\rho = 0$. It follows that

$$\sum_{|\rho|=2r} |\gamma_\rho|^2 = \sum_{|\mu|=r} |\gamma_{2\mu}|^2 = \sum_{|\mu|=r} |c_\mu|^2 |c_{-\mu}|^2. \tag{3}$$

Third, suppose that $\rho \in \mathbb{Z}^2, 0 < |\rho| < 2r$. Then, for

$$C_\rho = \{\mu \in \mathbb{Z}^2 : |\mu| = r, |\mu - \rho| = |\rho|\},$$

we have $|C_\rho| \leq 2$. If $|C_\rho| = 0$ then $\gamma_\rho = 0$. If $|C_\rho| = 1$ and $C_\rho = \{\mu\}$, then $\gamma_\rho = c_\mu \overline{c_{\mu-\rho}}$ and so $|\gamma_\rho|^2 = |c_\mu|^2 |c_{\mu-\rho}|^2$. If $|C_\rho| = 2$ and $C_\rho = \{\mu, m\}$, then $\gamma_\rho = c_\mu \overline{c_{\mu-\rho}} + c_m \overline{c_{m-\rho}}$ and so

$$|\gamma_\rho|^2 \leq 2|c_\mu|^2 |c_{\mu-\rho}|^2 + 2|c_m|^2 |c_{m-\rho}|^2.$$

It follows that

$$\sum_{0<|\rho|<2r} |\gamma_\rho|^2 \leq 4 \sum_{|\mu|=r, |\nu|=r, 0<|\mu-\nu|<2r} |c_\mu|^2 |c_\nu|^2.$$

Using (3) and then (2),

$$\begin{aligned}
\sum_{0<|\rho|\leq 2r} |\gamma_\rho|^2 &\leq 4 \sum_{|\mu|=r, |\nu|=r, 0<|\mu-\nu|<2r} |c_\mu|^2 |c_\nu|^2 + \sum_{|\mu|=r} |c_\mu|^2 |c_{-\mu}|^2 \\
&\leq 4 \sum_{|\mu|=r, |\nu|=r, 0<|\mu-\nu|<2r} |c_\mu|^2 |c_\nu|^2 + 4 \sum_{|\mu|=r} |c_\mu|^2 |c_{-\mu}|^2 \\
&\leq 4 \sum_{|\mu|=r, |\nu|=r} |c_\mu|^2 |c_\nu|^2 \\
&= 4 \left(\sum_{|\mu|=r} |c_\mu|^2 \right)^2 \\
&= 4.
\end{aligned}$$

Fourth, if $\rho \in \mathbb{Z}^2, |\rho| > 2r$ then $\gamma_\rho = 0$. Putting the above together, we have

$$\sum_{\rho \in \mathbb{Z}^2} |\gamma_\rho|^2 \leq 1 + 4 = 5.$$

Hence $\|c\|_4^4 \leq 5$, and therefore

$$|S| = \left| \int_{\mathbb{T}^2} f(x)\overline{c(x)}dx \right| \leq \int_{\mathbb{T}^2} |f(x)\overline{c(x)}|dx \leq \|f\|_{4/3} \|c\|_4 \leq \|f\|_{4/3} 5^{1/4},$$

proving the claim. □

2 Tensor products of functions

For $f_1 : X_1 \to \mathbb{C}$ and $f_2 : X_2 \to \mathbb{C}$, we define $f_1 \otimes f_2 : X_1 \times X_2 \to \mathbb{C}$ by

$$f_1 \otimes f_2(x_1, x_2) = f_1(x_1)f_2(x_2), \qquad (x_1, x_2) \in X_1 \times X_2.$$

For $f_1 \in L^1(\mathbb{T}^{d_1})$ and $f_2 \in L^1(\mathbb{T}^{d_2})$, it follows from Fubini's theorem that $f_1 \otimes f_2 \in L^1(\mathbb{T}^{d_1+d_2})$.

For $\xi_1 \in \mathbb{Z}^{d_1}$ and $\xi_2 \in \mathbb{Z}^{d_2}$, Fubini's theorem gives us

$$\begin{aligned}
\widehat{f_1 \otimes f_2}(\xi_1, \xi_2) &= \int_{\mathbb{T}^{d_1+d_2}} f_1 \otimes f_2(x_1, x_2)e^{-2\pi i(\xi_1,\xi_2)\cdot(x_1,x_2)}d\lambda_{d_1+d_2}(x_1, x_2) \\
&= \int_{\mathbb{T}^{d_1}} \left(\int_{\mathbb{T}^{d_2}} f_1 \otimes f_2(x_1, x_2)e^{-2\pi i(\xi_1,\xi_2)\cdot(x_1,x_2)}d\lambda_{d_2}(x_2) \right) d\lambda_{d_1}(x_1) \\
&= \int_{\mathbb{T}^{d_1}} f_1(x_1)e^{-2\pi i\xi_1\cdot x_1} \left(\int_{\mathbb{T}^{d_2}} f_2(x_2)e^{-2\pi i\xi_2\cdot x_2}d\lambda_{d_2}(x_2) \right) d\lambda_{d_1}(x_1) \\
&= \hat{f}_1(\xi_1)\hat{f}_2(\xi_2) \\
&= \hat{f}_1 \otimes \hat{f}_2(\xi_1, \xi_2),
\end{aligned}$$

showing that the Fourier transform of a tensor product is the tensor product of the Fourier transforms.

3 Approximate identities and Bernstein's inequality for $\mathbb{T}$

An **approximate identity** is a sequence k_N in $L^\infty(\mathbb{T}^d)$ such that (i) $\sup_N \|k_N\|_1 < \infty$, (ii) for each N,

$$\int_{\mathbb{T}^d} k_N(x)d\lambda_d(x) = 1,$$

and (iii) for each $0 < \delta < \frac{1}{2}$,

$$\lim_{n\to\infty} \int_{\delta \leq x \leq 1-\delta} |k_N(x)|d\lambda_d(x) = 0.$$

Suppose that k_N is an approximate identity. It is a fact that if $f \in C(\mathbb{T}^d)$ then $k_N * f \to f$ in $C(\mathbb{T}^d)$, if $1 \leq p < \infty$ and $f \in L^p(\mathbb{T}^d)$ then $k_N * f \to f$ in $L^p(\mathbb{T}^d)$, and if μ is a complex Borel measure on $\mathbb{T}^d$ then $k_N * \mu$ weak-* converges to μ.[2] (The Riesz representation theorem tells us that the Banach space $\mathcal{M}(\mathbb{T}^d) = rca(\mathbb{T}^d)$ of complex Borel measures on $\mathbb{T}^d$, with the total variation norm, is the dual space of the Banach space $C(\mathbb{T}^d)$.)

A **trigonometric polynomial** is a function $P : \mathbb{T}^d \to \mathbb{C}$ of the form

$$P(x) = \sum_{\xi \in \mathbb{Z}^d} a_\xi e^{2\pi i \xi \cdot x}, \qquad x \in \mathbb{T}^d$$

for which there is some $N \geq 0$ such that $a_\xi = 0$ whenever $|\xi|_\infty > N$. We say that P has **degree** N; thus, if P is a trigonometric polynomial of degree N then P is a trigonometric polynomial of degree M for each $M \geq N$.

For $f \in L^1(\mathbb{T})$, we define $S_N f \in C(\mathbb{T})$ by

$$(S_N f)(x) = \sum_{|j| \leq N} \hat{f}(j) e^{2\pi i j x}, \qquad x \in \mathbb{T}.$$

We define the **Dirichlet kernel** $D_N : \mathbb{T} \to \mathbb{C}$ by

$$D_N(x) = \sum_{|j| \leq N} e^{2\pi i j x}, \qquad x \in \mathbb{T},$$

which satisfies, for $f \in L^1(\mathbb{T})$,

$$D_N * f = S_N f.$$

We define the **Fejér kernel** $F_N \in C(\mathbb{T})$ by

$$F_N = \frac{1}{N+1} \sum_{n=0}^{N} D_n,$$

We can write the Fejér kernel as

$$F_N(x) = \sum_{|j| \leq N} \left(1 - \frac{|j|}{N+1}\right) e^{2\pi i j x} = \sum_{j \in \mathbb{Z}} \chi_{[-N,N]}(j) \left(1 - \frac{|j|}{N+1}\right) e^{2\pi i j x},$$

where χ_A is the indicator function of the set A. It is straightforward to prove that F_N is an approximate identity.

We define the d**-dimensional Fejér kernel** $F_{N,d} \in C(\mathbb{T}^d)$ by

$$F_{N,d} = \underbrace{F_N \otimes \cdots \otimes F_N}_{d}.$$

[2] Camil Muscalu and Wilhelm Schlag, *Classical and Multilinear Harmonic Analysis*, volume I, p. 10, Proposition 1.5.

We can write $F_{N,d}$ as

$$F_{N,d}(x) = \sum_{|\xi|_\infty \le N} \left(1 - \frac{|\xi_1|}{N+1}\right) \cdots \left(1 - \frac{|\xi_d|}{N+1}\right) e^{2\pi i \xi\cdot x}, \qquad x \in \mathbb{T}^d.$$

Using the fact that F_N is an approximate identity on $\mathbb{T}$, one proves that $F_{N,d}$ is an approximate identity on $\mathbb{T}^d$.

The following is **Bernstein's inequality for** $\mathbb{T}$.

Theorem 2 (Bernstein's inequality)**.** *If P is a trigonometric polynomial of degree N, then*

$$\|P'\|_\infty \le 4\pi N \|P\|_\infty .$$

Proof. Define

$$Q = ((e_{-N}P) * F_{N-1})e_N - ((e_N P) * F_{N-1})e_{-N}.$$

The Fourier transform of the first term on the right-hand side is, for $j \in \mathbb{Z}$,

$$\begin{aligned}
(e_{-N}\widehat{P * F}_{N-1}) * \widehat{e_N}(j) &= \sum_{k\in\mathbb{Z}} \widehat{e_{-N}P}(j-k)\widehat{F_{N-1}}(j-k)\widehat{e_N}(k) \\
&= \widehat{e_{-N}P}(j-N)\widehat{F_{N-1}}(j-N) \\
&= \widehat{P}(j)\widehat{F_{N-1}}(j-N),
\end{aligned}$$

and the Fourier transform of the second term is

$$\widehat{P}(j)\widehat{F_{N-1}}(j+N).$$

Therefore, for $j \in \mathbb{Z}$, using $\widehat{P} = \chi_{[-N,N]}\widehat{P}$,

$$\begin{aligned}
\widehat{Q}(j) &= \widehat{P}(j)\left(\widehat{F_{N-1}}(j-N) - \widehat{F_{N-1}}(j+N)\right) \\
&= \widehat{P}(j)\left(\chi_{[-N+1,N-1]}(j-N)\left(1 - \frac{|j-N|}{N}\right) - \chi_{[-N+1,N-1]}\left(1 - \frac{|j+N|}{N}\right)\right) \\
&= \widehat{P}(j)\Bigg(\chi_{[1,N]}(j)\left(1 + \frac{j-N}{N}\right) + \chi_{[N,2N-1]}(j)\left(1 - \frac{j-N}{N}\right) \\
&\quad - \chi_{[-2N+1,-N]}(j)\left(1 + \frac{j+N}{N}\right) - \chi_{[-N,-1]}(j)\left(1 - \frac{j+N}{N}\right)\Bigg) \\
&= \widehat{P}(j)\left(\chi_{[1,N]}(j)\left(1 + \frac{j-N}{N}\right) - \chi_{[-N,-1]}(j)\left(1 - \frac{j+N}{N}\right)\right) \\
&= \widehat{P}(j)\left(\frac{j}{N}\chi_{[1,N]}(j) + \frac{j}{N}\chi_{[-N,-1]}(j)\right) \\
&= \frac{j}{N}\widehat{P}(j).
\end{aligned}$$

On the other hand,

$$\widehat{P'}(j) = 2\pi i j \widehat{P}(j),$$

so that

$$P' = 2\pi i N Q,$$

i.e.

$$P' = 2\pi i N(((e_{-N}P) * F_{N-1})e_N - ((e_N P) * F_{N-1})e_{-N}).$$

Then, by Young's inequality,

$$\begin{aligned}
\|P'\|_\infty &= 2\pi N \left\|((e_{-N}P) * F_{N-1})e_N - ((e_N P) * F_{N-1})e_{-N}\right\|_\infty \\
&\leq 2\pi N \left\|((e_{-N}P) * F_{N-1})e_N\right\|_\infty + 2\pi N \left\|((e_N P) * F_{N-1})e_{-N}\right\|_\infty \\
&= 2\pi N \left\|(e_{-N}P) * F_{N-1}\right\|_\infty + 2\pi N \left\|(e_N P) * F_{N-1}\right\|_\infty \\
&\leq 2\pi N \left\|e_{-N}P\right\|_\infty \left\|F_{N-1}\right\|_1 + 2\pi N \left\|e_N P\right\|_\infty \left\|F_{N-1}\right\|_1 \\
&= 4\pi N \left\|P\right\|_\infty .
\end{aligned}$$

□