The theorem of F. and M. Riesz

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1 Totally ordered groups

Suppose that G is a locally compact abelian group and that $P \subset G$ is a **semigroup** (satisfies $P + P \subset P$) that is closed and satisfies $P \cap (-P) = \{0\}$ and $P \cup (-P) = G$. We define a **total order** on G by $x \leq y$ when $y - x \in P$. We verify that this is indeed a total order. (We remark that nowhere in this do we show the significance of P being closed; but in this note we shall be speaking about discrete abelian groups where any set is closed.)

If $x \leq y$ and $y \leq z$, then $y - x \in P$ and $z - y \in P$ and hence $z - x = (z-y) + (y-x) \in P + P \subset P$, showing that $x \leq z$, so $\leq$ is transitive. If $x \leq y$ and $y \leq x$ then $y - x \in P$ and $x - y \in P$, the latter of which is equivalent to $y - x = -(x-y) \in -P$, hence $y - x \in P \cap (-P)$, and then $P \cap (-P) = \{0\}$ implies that $y - x = 0$, i.e. $x = y$, so $\leq$ is antisymmetric. If $x, y \in P$ then $y - x$ is either 0, in which case $x = y$, or it is contained in one and only one of P and $-P$, and then respectively $x < y$ or $y < x$, showing that $\leq$ is total.

Moreover, the total order $\leq$ induced by the semigroup P is compatible with the group operation in G: if $x \leq y$ and $z \in G$, then $(y+z) - (x+z) = y - x \in P$, showing that $x + z \leq y + z$.

We say that G with the total order induced by P is a **totally ordered group**. We shall use the following lemma in the next section.[1]

Lemma 1. *Suppose that Γ is a discrete abelian group. Γ can be totally ordered if and only if $\gamma \in \Gamma$ having finite order implies that $\gamma = 0$.*

2 Functions of analytic type

If G is a compact abelian group, then G is connected if and only if $\gamma \in \widehat{G}$ having finite order implies that $\gamma = 0$.[2] Combined with Lemma 1, we get that a compact abelian group is connected if and only if its dual group can be ordered.

Suppose in the rest of this section that G is a connected compact abelian group, and let $\leq$ be a total order on $\widehat{G}$ induced by some semigroup. We say that a function $f \in L^1(G)$ is of **analytic type** if $\gamma < 0$ implies that $\hat{f}(\gamma) = 0$,

[1] Walter Rudin, *Fourier Analysis on Groups*, p. 194, Theorem 8.1.2.

[2] Walter Rudin, *Fourier Analysis on Groups*, p. 47, Theorem 2.5.6.

and we say that a measure $\mu \in M(G)$ is of analytic type if $\gamma < 0$ implies that $\hat{\mu}(\gamma) = 0$. (We denote by $M(G)$ the set of **regular complex Borel measures** on G.) For $1 \leq p \leq \infty$, we denote by $H^p(G)$ those elements of $L^p(G)$ that are of analytic type. We emphasize that the notion of a function or measure being of analytic type depends on the total order $\leq$ on $\widehat{G}$.

We remind ourselves that when $\mathscr{M}$ is a σ-algebra on a set X and μ is a measure on $\mathscr{M}$, if $A \in \mathscr{M}$ and $\mu(E) = \mu(A \cap E)$ for all $E \in \mathscr{M}$ then we say that μ is **concentrated on** A. Measures λ, μ on $\mathscr{M}$ are said to be **mutually singular** if they are concentrated on disjoint sets.

Let m be the Haar measure on G such that $m(G) = 1$, and suppose that σ is a positive element of $M(G)$. The **Lebesgue decomposition** tells us that there is a unique pair of finite Borel measures σ_s and σ_a on G such that (i) $\sigma = \sigma_s + \sigma_a$, (ii) σ_a is absolutely continuous with respect to m, and (iii) σ_s and m are mutually singular. Then the **Radon-Nikodym theorem** tells us that there is a unique nonnegative $w \in L^1(m)$ such that $d\sigma_a = w dm$. Thus,

$$d\sigma = d\sigma_s + w dm.$$

We define Ω to be the set of all trigonometric polynomials Q on G such that $\hat{Q}(\gamma) = 0$ for $\gamma \leq 0$. We also define $K = \{1 + Q : Q \in \Omega\}$. $K \subset L^2(\sigma)$, and we denote by $\overline{K}$ its closure in the Hilbert space $L^2(\sigma)$.

Lemma 2. *$\overline{K}$ is a convex set.*

Proof. Let $f, g \in K$ be distinct and let $0 \leq t \leq 1$. There are $P_n, Q_n \in \Omega$ such that $1 + P_n \to f$ and $1 + Q_n \to g$, and

$$(1-t)f + tg = \lim_{n \to \infty} ((1-t)(1+P_n) + t(1+Q_n)) = \lim_{n \to \infty} (1 + (1-t)P_n + tQ_n).$$

For each n, $(1-t)P_n + tQ_n \in \Omega$, so we have written $(1-t)f + tg$ as a limit of elements of K, showing that $(1-t)f + tg \in \overline{K}$ and hence that $\overline{K}$ is convex. $\square$

As $\overline{K}$ is a closed convex set in the Hilbert space $L^2(\sigma)$, there is a unique $\phi \in \overline{K}$ such that $d(0, \overline{K}) = \|0 - \phi\|$ (namely, that attains the infimum of the distance of elements of $\overline{K}$ to the origin), which we can write as

$$\|\phi\| = \inf_{Q \in \Omega} \|1 + Q\|.$$

ϕ is the unique element of $\overline{K}$ such that

$$\langle \phi, \psi - \phi \rangle = 0, \qquad \psi \in \overline{K}.$$

The following lemma establishes properties of ϕ.[3]

Lemma 3. *1. $\phi = 0$ almost everywhere with respect to σ_s.*

2. $\phi w \in L^2(m)$ and $|\phi|^2 w = \|\phi\|^2$ almost everywhere with respect to m.

[3]Walter Rudin, *Fourier Analysis on Groups*, p. 199, Lemma 8.2.2.

3. If $\|\phi\| > 0$ and $h = \frac{1}{\phi}$, then $h \in H^2(m)$ and $\hat{h}(0) = 1$.

Proof. We write $c = \|\phi\|$. Let $1 + Q_n \in K$ such that $1 + Q_n \to \phi$. If $g \in L^2(\sigma)$ and $\phi + g \in \overline{K}$, then $\langle \phi, (\phi + g) - \phi \rangle = 0$, i.e. $\langle \phi, g \rangle = 0$. Let $\gamma > 0$. On the one hand, $\gamma \in \Omega$ so $\phi + \gamma = \lim_{n\to\infty} 1 + (Q_n + \gamma) \in \overline{K}$, hence $\langle \phi, \gamma \rangle = 0$ and so $\langle \gamma, \phi \rangle = 0$. On the other hand, define $g = \phi\gamma$, which satisfies

$$\phi + g = \phi(1+\gamma) = \lim_{n\to\infty}(1+Q_n)(1+\gamma) = \lim_{n\to\infty} 1 + \gamma + Q_n + Q_n\gamma,$$

and because $\gamma > 0$, each term of $\gamma + Q_n + Q_n\gamma$ belongs to Ω, showing that $\phi + g \in \overline{K}$, from which we get $\langle \phi, g \rangle = 0$ and so $\langle g, \phi \rangle = 0$. We have proved that

$$\int_G \langle x, \gamma \rangle \overline{\phi(x)} d\sigma(x) = 0, \qquad \gamma > 0, \tag{1}$$

and

$$\int_G \langle x, \gamma \rangle |\phi(x)|^2 d\sigma(x) = 0, \qquad \gamma > 0. \tag{2}$$

Taking the complex conjugate of (2) gives

$$\int_G \langle x, \gamma \rangle |\phi(x)|^2 d\sigma(x) = 0, \qquad \gamma < 0.$$

Defining $d\lambda = |\phi|^2 d\sigma$ we have $\lambda \in M(G)$. The above and (2) give

$$\hat{\lambda}(\gamma) = 0, \qquad \gamma \neq 0.$$

As well,

$$\hat{\lambda}(0) = \int_G |\phi|^2 d\sigma = c^2.$$

Because $\lambda \in M(G)$ and $\hat{\lambda} \in L^1(\widehat{G})$, there is some $f \in L^1(G)$ such that $d\lambda = f dm$, defined by

$$f(x) = \int_{\widehat{G}} \hat{\lambda}(\gamma) \langle x, \gamma \rangle dm_{\widehat{G}}(\gamma), \qquad \gamma \in \widehat{G},$$

where $m_{\widehat{G}}$ is the Haar measure on $\widehat{G}$ that assigns measure 1 to each singleton.[4] That is, $d\lambda = f dm$ where $f(x) = c^2 m_{\widehat{G}}(\{0\}) = c^2$, hence $d\lambda = c^2 dm$. Combined with $d\lambda = |\phi|^2 d\sigma$ we get

$$|\phi|^2 d\sigma = c^2 dm.$$

Therefore $|\phi|^2 d\sigma$ is absolutely continuous with respect to m, and because $|\phi|^2 d\sigma = |\phi|^2 d\sigma_s + |\phi|^2 w dm$, it follows that $|\phi|^2 d\sigma_s = 0$, that is, that $\phi(x) = 0$ for σ_s-almost all $x \in G$, proving the first claim. Furthermore, $|\phi|^2 d\sigma = |\phi|^2 w dm$ and using $|\phi|^2 d\sigma = c^2 dm$ we get $|\phi(x)|^2 w(x) = c^2$ for m-almost all $x \in G$. Because $w \in L^1(m)$ and $|\phi w|^2 = c^2 w$, we get $\phi w \in L^2(m)$, proving the second claim.

[4] Walter Rudin, *Fourier Analysis on Groups*, p. 30.

So far we have not supposed that $c > 0$. If indeed $c > 0$, then $|h|^2 = |\phi|^{-2} = c^{-2}w$, giving $h \in L^2(m)$. For $\gamma \in \widehat{G}$,

$$\begin{aligned} \int_G h(x)\langle x, \gamma\rangle dm(x) &= \int_G |\phi(x)|^{-2}\overline{\phi(x)}\langle x, \gamma\rangle dm(x) \\ &= c^{-2}\int_G \langle x, \gamma\rangle \overline{\phi(x)} w(x) dm(x) \\ &= c^{-2}\int_G \langle x, \gamma\rangle \overline{\phi(x)} d\sigma(x). \end{aligned}$$

This and (1) yield

$$\int_G h(x)\langle x, \gamma\rangle dm(x) = 0, \qquad \gamma > 0,$$

in other words,

$$\hat{h}(\gamma) = 0, \qquad \gamma < 0,$$

namely, h is of analytic type, i.e. $h \in H^2(m)$. Moreover, for each $n \in \mathbb{N}$ we check that $Q_n + \phi \in \overline{K}$ and hence that $\langle 1 + Q_n, \phi\rangle = \langle 1, \phi\rangle$, giving

$$c^2\hat{h}(0) = \int_G \overline{\phi} d\sigma = \int_G (1 + Q_n)\overline{\phi} d\sigma.$$

This is true for all $n \in \mathbb{N}$, so we obtain

$$c^2\hat{h}(0) = \int_G |\phi|^2 d\sigma = \|\phi\|^2 = c^2,$$

i.e. $\hat{h}(0) = 1$, proving the third claim. □

The above lemma is used to prove the following theorem.[5] The proof of this theorem in Rudin is not long, but I don't understand the first step in his proof so I have not attempted to write it out.

Theorem 4. *Suppose that G is a connected compact abelian group and that $\mu \in M(G)$ is of analytic type. If the Lebesgue decomposition of μ is*

$$d\mu = d\mu_s + f dm,$$

where μ_s and m are mutually singular and $f \in L^1(m)$, then $\mu_s \in M(G)$ is of analytic type and f is of analytic type, and $\hat{\mu}_s(0) = 0$.

3 The theorem of F. and M. Riesz

We are now equipped to prove the theorem of F. and M. Riesz.[6]

[5] Walter Rudin, *Fourier Analysis on Groups*, p. 200, Theorem 8.2.3.
[6] Walter Rudin, *Fourier Analysis on Groups*, p. 201, §8.2.4.

Theorem 5 (F. and M. Riesz)**.** *If $\mu \in M(\mathbb{T})$ and $\hat{\mu}(n) = 0$ for every negative integer n, then μ is absolutely continuous with respect to Haar measure.*

Proof. Write $d\mu = d\mu_s + f dm$, where μ_s and m are mutually singular and $f \in L^1(m)$. Theorem 4 tells us that μ_s is of analytic type, i.e. $\hat{\mu}_s(n) = 0$ for $n < 0$, and that $\hat{\mu}_s(0) = 0$. Therefore, if $\mu_s \neq 0$ then there is a minimal positive integer n_0 for which $\hat{\mu}_s(n_0) \neq 0$. Defining $\hat{\lambda}(n) = \hat{\mu}_s(n_0 + n)$, we get that $\lambda \in M(\mathbb{T})$ and that λ and m are mutually singular. But $\hat{\lambda}(n) = \hat{\mu}_s(n_0 + n) = 0$ for $n < 0$, so λ is of analytic type, and therefore Theorem 4 says that $\hat{\mu}_s(n_0) = \hat{\lambda}(0) = 0$ (because λ and m are mutually singular), a contradiction. Hence $\hat{\mu}_s(n) = 0$ for all $n \in \mathbb{Z}$, which implies that $\mu_s = 0$. But this means that μ is absolutely continuous with respect to m, completing the proof. □