1st coefficient $= 1$;

2d $= \frac{10}{1} = 10$; 7th $= 252 \cdot \frac{5}{6} = 210$;
3d $= 10 \cdot \frac{9}{2} = 45$; 8th $= 210 \cdot \frac{4}{7} = 120$;
4th $= 45 \cdot \frac{8}{3} = 120$; 9th $= 120 \cdot \frac{3}{8} = 45$;
5th $= 120 \cdot \frac{7}{4} = 210$; 10th $= 45 \cdot \frac{2}{9} = 10$;
6th $= 210 \cdot \frac{6}{5} = 252$; and 11th $= 10 \cdot \frac{1}{10} = 1$.

351. We may also write these fractions as they are, without computing their value; and in this manner it is easy to express any power of $a + b$. Thus, $(a + b)^{100} = a^{100} + \frac{100}{1} \cdot a^{99}b + \frac{100 \cdot 99}{1 \cdot 2} + a^{98}b^2 + \frac{100 \cdot 99 \cdot 98}{1 \cdot 2 \cdot 3} a^{97}b^3 + \frac{100 \cdot 99 \cdot 98 \cdot 97}{1 \cdot 2 \cdot 3 \cdot 4} a^{96}b^4 +$, &c.* Whence the law of the succeeding terms may be easily deduced.

CHAP. XI.

Of the Transposition *of the* Letters, *on which the demonstration of the preceding* Rule *is founded.*

352. If we trace back the origin of the coefficients which we have been considering, we shall find, that each term is presented, as many times as it is possible to transpose the letters, of which that term is composed; or, to express the same thing differently, the coefficient of each term is equal to the number of transpositions which the letters composing that term admit of. In the second power, for example, the term ab is taken twice, that is to say, its coefficient is 2; and in fact we may change the order of the letters which compose that term twice, since we may write ab and ba.

* Or, which is a more general mode of expression,

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1} b + \frac{n \cdot (n-1)}{1 \cdot 2} a^{n-2} b^2 + \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3} a^{n-3} b^3 + \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4} b^4 \text{ \&c. } - - \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) - - - - 1}{1 \cdot 2 \cdot 3 \cdot \quad 4 - - - - - n}.$$

This elegant theorem for the involution of a compound quantity of two terms, evidently includes all powers whatever; and we shall afterwards shew how the same may be applied to the extraction of roots.

The term *aa*, on the contrary, is found only once, and here the order of the letters can undergo no change, or transposition. In the third power of $a + b$, the term *aab* may be written in three different ways; thus, *aab*, *aba*, *baa*; the coefficient therefore is 3. In the fourth power, the term a^3b or *aaab* admits of four different arrangements, *aaab*, *aaba*, *abaa*, *baaa*; and consequently the coefficient is 4. The term *aabb* admits of six transpositions, *aabb*, *abba*, *baba*, *abab*, *bbaa*, *baab*, and its coefficient is 6. It is the same in all other cases.

353. In fact, if we consider that the fourth power, for example, of any root consisting of more than two terms, as $(a + b + c + d)^4$, is found by the multiplication of the four factors, $(a + b + c + d)$ $(a + b + c + d)$ $(a + b + c + d)$ $(a + b + c + d)$, we readily see, that each letter of the first factor must be multiplied by each letter of the second, then by each letter of the third, and, lastly, by each letter of the fourth. So that every term is not only composed of four letters, but it also presents itself, or enters into the sum, as many times as those letters can be differently arranged with respect to each other; and hence arises its coefficient.

354. It is therefore of great importance to know, in how many different ways a given number of letters may be arranged; but, in this inquiry, we must particularly consider, whether the letters in question are the same, or different: for when they are the same, there can be no transposition of them; and for this reason the simple powers, as a^2, a^3, a^4, &c. have all unity for their coefficients.

355. Let us first suppose all the letters different; and, beginning with the simplest case of two letters, or *ab*, we immediately discover that two transpositions may take place, namely, *ab* and *ba*.

If we have three letters, *abc*, to consider, we observe that each of the three may take the first place, while the two others will admit of two transpositions; thus, if *a* be the first letter, we have two arrangements *abc*, *acb*; if *b* be in the first place, we have the arrangements *bac*, *bca*; lastly, if *c* occupy the first place, we have also two arrangements, namely, *cab*, *cba*; consequently the whole number of arrangements is $3 \times 2 = 6$.

If there be four letters *abcd*, each may occupy the first place; and in every case the three others may form six different arrangements, as we have just seen; therefore the whole number of transpositions is $4 \times 6 = 24 = 4 \times 3 \times 2 \times 1$.

If we have five letters, *abcde*, each of the five may be the first, and the four others will admit of twenty-four transpositions; so that the whole number of transpositions will be $5 \times 24 = 120 = 5 \times 4 \times 3 \times 2 \times 1$.

356. Consequently, however great the number of letters may be, it is evident, provided they are all different, that we may easily determine the number of transpositions, and that we may for this purpose make use of the following Table:

Number of Letters.	Number of Transpositions.
1	$1 = 1$.
2	$2 . 1 = 2$.
3	$3 . 2 . 1 = 6$.
4	$4 . 3 . 2 . 1 = 24$.
5	$5 . 4 . 3 . 2 . 1 = 120$.
6	$6 . 5 . 4 . 3 . 2 . 1 = 720$.
7	$7 . 6 . 5 . 4 . 3 . 2 . 1 = 5040$.
8	$8 . 7 . 6 . 5 . 4 . 3 . 2 . 1 = 40320$.
9	$9 . 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1 = 362880$.
10	$10 . 9 . 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1 = 3628800$.

357. But, as we have intimated, the numbers in this Table can be made use of only when all the letters are different; for if two or more of them are alike, the number of transpositions becomes much less; and if all the letters are the same, we have only one arrangement: we shall therefore now shew how the numbers in the Table are to be diminished, according to the number of letters that are alike.

358. When two letters are given, and those letters are the same, the two arrangements are reduced to one, and consequently the number, which we have found above, is reduced to the half; that is to say, it must be divided by 2. If we have three letters alike, the six transpositions are reduced to one; whence it follows that the numbers in the Table must be divided by $6 = 3 . 2 . 1$; and, for the same reason, if four letters are alike, we must divide the numbers found by 24, or $4 . 3 . 2 . 1$, &c.

It is easy therefore to find how many transpositions the letters *aaabbc*, for example, may undergo. They are in number 6, and consequently, if they were all different, they would admit of $6 . 5 . 4 . 3 . 2 . 1$ transpositions; but since a is found thrice in those letters, we must divide that number of transpositions by $3 . 2 . 1$; and since b occurs twice, we must again divide it by $2 . 1$: the number of transpositions required will therefore be $\frac{6 . 5 . 4 . 3 . 2 . 1}{3 . 2 . 1 . 2 . 1} = 5 . 4 . 3 = 60$.

359. We may now readily determine the coefficients of all the terms of any power; as for example of the seventh power $(a + b)^7$.

The first term is a^7, which occurs only once; and as all the other terms have each seven letters, it follows that the number of transpositions for each term would be 7 . 6 . 5 . 4 . 3 . 2 . 1, if all the letters were different; but since in the second term, a^6b, we find six letters alike, we must divide the above product by 6 . 5 . 4 . 3 . 2 . 1, whence it follows that the coefficient is $\frac{7 . 6 . 5 . 4 . 3 . 2 . 1}{6 . 5 . 4 . 3 . 2 . 1} = \frac{7}{1}$.

In the third term, a^5b^2, we find the same letter a five times, and the same letter b twice; we must therefore divide that number first by 5 . 4 . 3 . 2 . 1, and then by 2 . 1; whence results the coefficient $\frac{7 . 6 . 5 . 4 . 3 . 2 . 1}{5 . 4 . 3 . 2 . 1 . 2 . 1} = \frac{7 . 6}{1 . 2}$.

The fourth term a^4b^3 contains the letter a four times, and the letter b thrice; consequently, the whole number of the transpositions of the seven letters, must be divided, in the first place, by 4 . 3 . 2 . 1, and, secondly, by 3 . 2 . 1, and the coefficient becomes $= \frac{7 . 6 . 5 . 4 . 3 . 2 . 1}{4 . 3 . 2 . 1 . 3 . 2 . 1} = \frac{7 . 6 . 5}{1 . 2 . 3}$.

In the same manner, we find $\frac{7 . 6 . 5 . 4}{1 . 2 . 3 . 4}$ for the coefficient of the fifth term, and so of the rest; by which the rule before given is demonstrated *.

360. These considerations carry us farther, and shew us

* From the *Theory of Combinations,* also, are frequently deduced the rules that have just been considered for determining the coefficients of terms of the power of a binomial; and this is perhaps attended with some advantage, as the whole is then reduced to a single formula.

In order to perceive the difference between *permutations* and *combinations,* it may be observed, that in the former we inquire in how many different ways the letters, which compose a certain formula, may change places; whereas, in combinations it is only necessary to know how many times these letters may be taken or multiplied together, one by one, two by two, three by three, &c.

also how to find all the powers of roots composed of more than two terms*. We shall apply them to the third power of $a + b + c$; the terms of which must be formed by all the possible combinations of three letters, each term having for its coefficient the number of its transpositions, as shewn, Art. 352.

Here, without performing the multiplication, the third power of $(a + b + c)$ will be, $a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2 + 3bc^2 + c^3$.

Suppose $a = 1$, $b = 1$, $c = 1$, the cube of $1 + 1 + 1$, or of 3, will be $1+3+3+3+6+3+1+3+3+1=27$;

Let us take the formula *abc*; here we know that the letters which compose it admit of six permutations, namely *abc, acb, bac, bca, cab, cba*: but as for combinations, it is evident that by taking these three letters one by one, we have three combinations, namely, *a*, *b*, and *c*; if two by two, we have three combinations, *ab*, *ac*, and *bc*; lastly, if we take them three by three, we have only the single combination *abc*.

Now, in the same manner as we prove that n different things admit of $1 \times 2 \times 3 \times 4 \text{--} n$ different permutations, and that if r of these n things are equal, the number of permutations is $\frac{1 \times 2 \times 3 \times 4 \text{--} n}{1 \times 2 \times 3 \times \text{--} r}$; so likewise we prove that n things may be taken r by r, $\frac{n \times (n1) - \times (n-2) \text{---} (n-r+1)}{1 \times 2 \times 3 \text{--} r}$ number of times; or that we may take r of these n things in so many different ways. Hence, if we call n the exponent of the power to which we wish to raise the binomial $a + b$, and r the exponent of the letter b in any term, the coefficient of that term is always expressed by the formula $\frac{n \times (n-1) \times (n-2) \text{---} (n-r+1)}{1 \times 2 \times 3 \text{----} r}$. Thus, in the example, article 359, where $n = 7$, we have a^5b^2 for the third term, the exponent $r = 2$, and consequently the coefficient $= \frac{7 \times 6}{1 \times 2}$; for the fourth term we have $r = 3$, and the coefficient $= \frac{7 \times 6 \times 5}{1 \times 2 \times 3}$, and so on; which are evidently the same results as the permutations.

For complete and extensive treatises on the theory of combinations, we are indebted to *Frenicle*, *De Montmort*, *James Bernoulli*, &c. The two last have investigated this theory, with a view to its great utility in the calculation of probabilities. F. T.

* Roots, or quantities, composed of more than two terms, are called *polynomials*, in order to distinguish them from *binomials*, or quantities composed of two terms. F. T.

which result is accurate, and confirms the rule. But if we had supposed $a = 1$, $b = 1$, and $c = -1$, we should have found for the cube of $1 + 1 - 1$, that is of 1,

$1 + 3 - 3 + 3 - 6 + 3 + 1 - 3 + 3 - 1 = 1$, which is a still farther confirmation of the rule.

CHAP. XII.

Of the Expression *of* Irrational Powers *by* Infinite Series.

361. As we have shewn the method of finding any power of the root $a + b$, however great the exponent may be, we are able to express, generally, the power of $a + b$, whose exponent is undetermined; for it is evident that if we represent that exponent by n, we shall have by the rule already given (Art. 348 and the following):

$$(a + b)^n = a^n + \frac{n}{1} a^{-1}b + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2}b^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 + \&c.$$

362. If the same power of the root $a - b$ were required, we need only change the signs of the second, fourth, sixth, &c. terms, and should have

$$(a - b)^n = a^n - \frac{n}{1} a^{n-1}b + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2}b^2 - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 - \&c.$$

363. These formulas are remarkably useful, since they serve also to express all kinds of radicals; for we have shewn that all irrational quantities may assume the form of powers whose exponents are fractional, and that $\sqrt[2]{a} = a^{\frac{1}{2}}$, $\sqrt[3]{a} = a^{\frac{1}{3}}$, and $\sqrt[4]{a} = a^{\frac{1}{4}}$, &c.: we have, therefore,

$$\sqrt[2]{(a + b)} = (a + b)^{\frac{1}{2}};\ \sqrt[3]{(a + b)} = (a + b)^{\frac{1}{3}};$$
$$\text{and } \sqrt[4]{(a + b)} = (a + b)^{\frac{1}{4}}, \&c.$$

Consequently, if we wish to find the square root of $a + b$, we have only to substitute for the exponent n the fraction $\frac{1}{2}$, in the general formula, Art. 361, and we shall have first, for the coefficients,