CHAP. VII.

Of Fractions *in general.*

68. When a number, as 7, for instance, is said not to be divisible by another number, let us suppose by 3, this only means, that the quotient cannot be expressed by an integer number; but it must not by any means be thought that it is impossible to form an idea of that quotient. Only imagine a line of 7 feet in length; nobody can doubt the possibility of dividing this line into 3 equal parts, and of forming a notion of the length of one of those parts.

69. Since therefore we may form a precise idea of the quotient obtained in similar cases, though that quotient may not be an integer number, this leads us to consider a particular species of numbers, called *fractions*, or *broken numbers;* of which the instance adduced furnishes an illustration. For if we have to divide 7 by 3, we easily conceive the quotient which should result, and express it by $\frac{7}{3}$; placing the divisor under the dividend, and separating the two numbers by a stroke, or line.

70. So, in general, when the number a is to be divided by the number b, we represent the quotient by $\frac{a}{b}$, and call this form of expression *a fraction*. We cannot therefore give a better idea of a fraction $\frac{a}{b}$, than by saying that it expresses the quotient resulting from the division of the upper number by the lower. We must remember also, that in all fractions the lower number is called the *denominator*, and that above the line the *numerator*.

71. In the above fraction $\frac{7}{3}$, which we read *seven thirds*, 7 is the numerator, and 3 the denominator. We must also read $\frac{2}{3}$, two thirds; $\frac{3}{4}$, three fourths; $\frac{3}{8}$, three eighths; $\frac{12}{100}$, twelve hundredths; and $\frac{1}{2}$, one half, &c.

72. In order to obtain a more perfect knowledge of the nature of fractions, we shall begin by considering the case in which the numerator is equal to the denominator, as in $\frac{a}{a}$. Now, since this expresses the quotient obtained by dividing a by a, it is evident that this quotient is exactly unity, and that consequently the fraction $\frac{a}{a}$ is of the same

value as 1, or one integer; for the same reason, all the following fractions,

$$\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}, \&c.$$

are equal to one another, each being equal to 1, or one integer.

73. We have seen that a fraction whose numerator is equal to the denominator, is equal to unity. All fractions therefore whose numerators are less than the denominators, have a value less than unity: for if I have a number to divide by another, which is greater than itself, the result must necessarily be less than 1. If we cut a line, for example, two feet long, into three parts, one of those parts will undoubtedly be shorter than a foot: it is evident then, that $\frac{2}{3}$ is less than 1, for the same reason; that is, the numerator 2 is less than the denominator 3.

74. If the numerator, on the contrary, be greater than the denominator, the value of the fraction is greater than unity. Thus $\frac{3}{2}$ is greater than 1, for $\frac{3}{2}$ is equal to $\frac{2}{2}$ together with $\frac{1}{2}$. Now $\frac{2}{2}$ is exactly 1; consequently $\frac{3}{2}$ is equal to $1 + \frac{1}{2}$, that is, to an integer and a half. In the same manner, $\frac{4}{3}$ is equal to $1\frac{1}{3}$, $\frac{5}{3}$ to $1\frac{2}{3}$, and $\frac{7}{3}$ to $2\frac{1}{3}$. And, in general, it is sufficient in such cases to divide the upper number by the lower, and to add to the quotient a fraction, having the remainder for the numerator, and the divisor for the denominator. If the given fraction, for example, were $\frac{43}{12}$, we should have for the quotient 3, and 7 for the remainder; whence we should conclude that $\frac{43}{12}$ is the same as $3\frac{7}{12}$.

75. Thus we see how fractions, whose numerators are greater than the denominators, are resolved into two members; one of which is an integer, and the other a fractional number, having the numerator less than the denominator. Such fractions as contain one or more integers, are called *improper fractions*, to distinguish them from fractions properly so called, which having the numerator less than the denominator, are less than unity, or than an integer.

76. The nature of fractions is frequently considered in another way, which may throw additional light on the subject. If, for example, we consider the fraction $\frac{3}{4}$, it is evident that it is three times greater than $\frac{1}{4}$. Now, this fraction $\frac{1}{4}$ means, that if we divide 1 into 4 equal parts, this will be the value of one of those parts; it is obvious then, that by taking 3 of those parts we shall have the value of the fraction $\frac{3}{4}$.

In the same manner we may consider every other fraction; for example, $\frac{7}{12}$; if we divide unity into 12 equal parts, 7 of those parts will be equal to the fraction proposed.

77. From this manner of considering fractions, the expressions *numerator* and *denominator* are derived. For, as in the preceding fraction $\frac{7}{12}$, the number under the line shews that 12 is the number of parts into which unity is to be divided; and as it may be said to denote, or name, the parts, it has not improperly been called the *denominator*.

Farther, as the upper number, viz. 7, shews that, in order to have the value of the fraction, we must take, or collect, 7 of those parts, and therefore may be said to reckon or number them, it has been thought proper to call the number above the line the *numerator*.

78. As it is easy to understand what $\frac{3}{4}$ is, when we know the signification of $\frac{1}{4}$, we may consider the fractions whose numerator is unity, as the foundation of all others. Such are the fractions,

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \tfrac{1}{8}, \tfrac{1}{9}, \tfrac{1}{10}, \tfrac{1}{11}, \tfrac{1}{12}, \text{\&c.}$$

and it is observable that these fractions go on continually diminishing: for the more you divide an integer, or the greater the number of parts into which you distribute it, the less does each of those parts become. Thus, $\frac{1}{100}$ is less than $\frac{1}{10}$; $\frac{1}{1000}$ is less than $\frac{1}{100}$; and $\frac{1}{10000}$ is less than $\frac{1}{1000}$, &c.

79. As we have seen that the more we increase the denominator of such fractions the less their values become, it may be asked, whether it is not possible to make the denominator so great that the fraction shall be reduced to nothing? I answer, no; for into whatever number of parts unity (the length of a foot, for instance) is divided; let those parts be ever so small, they will still preserve a certain magnitude, and therefore can never be absolutely reduced to nothing.

80. It is true, if we divide the length of a foot into 1000 parts, those parts will not easily fall under the cognisance of our senses; but view them through a good microscope, and each of them will appear large enough to be still subdivided into 100 parts, and more.

At present, however, we have nothing to do with what depends on ourselves, or with what we are really capable of performing, and what our eyes can perceive; the question is rather what is possible in itself: and, in this sense, it is certain, that however great we suppose the denominator, the fraction will never entirely vanish, or become equal to 0.

81. We can never therefore arrive completely at 0, or nothing, however great the denominator may be; and, consequently, as those fractions must always preserve a certain quantity, we may continue the series of fractions in the 78th article without interruption. This circumstance has in-

troduced the expression, that the denominator must be *infinite*, or infinitely great, in order that the fraction may be reduced to 0, or to nothing; hence the word *infinite* in reality signifies here, that we can never arrive at the end of the series of the above-mentioned *fractions*.

82. To express this idea, according to the sense of it above-mentioned, we make use of the sign ∞, which consequently indicates a number infinitely great; and we may therefore say, that this fraction $\frac{1}{\infty}$ is in reality nothing; because a fraction cannot be reduced to nothing, until the denominator has been increased to *infinity*.

83. It is the more necessary to pay attention to this idea of infinity, as it is derived from the first elements of our knowledge, and as it will be of the greatest importance in the following part of this treatise.

We may here deduce from it a few consequences that are extremely curious, and worthy of attention. The fraction $\frac{1}{\infty}$ represents the quotient resulting from the division of the dividend 1 by the divisor ∞. Now, we know, that if we divide the dividend 1 by the quotient $\frac{1}{\infty}$, which is equal to nothing, we obtain again the divisor ∞: hence we acquire a new idea of infinity; and learn that it arises from the division of 1 by 0; so that we are thence authorised in saying, that 1 divided by 0 expresses a number infinitely great, or ∞.

84. It may be necessary also, in this place, to correct the mistake of those who assert, that a number infinitely great is not susceptible of increase. This opinion is inconsistent with the just principles which we have laid down; for $\frac{1}{0}$ signifying a number infinitely great, and $\frac{2}{0}$ being incontestably the double of $\frac{1}{0}$, it is evident that a number, though infinitely great, may still become twice, thrice, or any number of times greater *.

* There appears to be a fallacy in this reasoning, which consists in taking the sign of infinity for infinity itself; and applying the property of fractions in general to a fractional expression, whose denominator bears no assignable relation to unity. It is certain, that infinity may be represented by a series of units (that is, by $\frac{1}{0} = \frac{1}{1-1} = 1 + 1 + 1$, &c.) or by a series of numbers increasing in any given ratio. Now, though any definite part of one infinite series may be the half, the third, &c. of a definite part of another, yet still that part bears no proportion to the whole, and the series can only be said, in that case, to go on to infinity in a different ratio. But, farther, $\frac{2}{0}$, or any other nu-